NATIONAL MATHEMATICS MAGAZINE

VOL. XX BATON ROUGE, LA., November, 1945 No. 2

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VOL. XX — BATON ROUGE, LA., November, 1945 — No. 2

Entered as second-class matter at Baton Rouge, Louisiana.
Published monthly excepting June, July, August, and September, by S. T. SANDERS.

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A Brace of Questions

(1) Is mathematics, like religion, most real only when it is *applied?*

(2) If mathematics is an *aristocratic* science, that is, reserved for a minority of selected minds, how can it be perfectly integrated with the schools of a democracy?

(3) Is the *democratic* character of mathematical science proved by the fact of its multiform applications in an industrial age and by the prospect of its vastly increased uses to come in a so-called atomic age?

Or, do these very conditions bear witness to its *aristocratic* character, inasmuch as they point to the need of selective minds to solve an increasing horde of specialized problems?

(4) Waiving as academic these questions concerning a democratic or an aristocratic quality of mathematics, let us ask:

Despite an age-to-age expansion of the science which more than all other sciences embodies the laws of perfect thinking, is it true, or does it merely seem to be true, that its influence on the thinking of the average-man-on-the-street is lessening?

S. T. Sanders.

Differential Equations with Quadrilateral Envelope---Cuspidal and Nodal Loci

By ARCHIBALD HENDERSON
University of North Carolina

1. In treatises on advanced calculus and differential equations, simple types of differential equations having four straight lines as envelope occur occasionally. These examples usually present special cases: square, rectangle, and parallelogram. No general method of solution appears to have been developed for the case of the complete quadrilateral. The solution of special cases is frequently effected by means of substitutions which reduce the given differential equation to Clairaut's form

$$y=px+f(p)\cdots\cdots\left(p=\frac{dy}{dx}\right). \tag{1}$$

The question naturally arises as to whether the solution of the general problem may not always be effected by means of some discoverable substitution, reducing the differential equation to Clairaut's form. So far as is known to the writer, no sure-fire method for disengaging such substitution, without advance knowledge of the general solution, has been developed. Another method of approach, which always effects a general solution, is developed in the present paper.

2. The converse of the problem posed above is readily handled by means of Clairaut's form of differential equation. Suppose the general solution of a differential equation, which remains to be determined, is given in the form

$$\phi(x,y)c^2+\psi(x,y)c+\chi(x,y)=0 \tag{2}$$

where ϕ,ψ,χ are rational, integral expressions of the second degree in x and y. Dividing through by $\phi(x,y)$ and differentiating, we have

$$c=-\left(\frac{\phi\cdot\chi'-\chi\cdot\phi'}{\phi\cdot\psi'-\psi\cdot\phi'}\right). \tag{3}$$

Eliminating c between (2) and (3), we have

$$\left(\frac{\phi\cdot\chi'-\chi\cdot\phi'}{\phi\cdot\psi'-\psi\cdot\phi'}\right)^2-\frac{\psi}{\phi}\left(\frac{\phi\cdot\chi'-\chi\cdot\phi'}{\phi\cdot\psi'-\psi\cdot\phi'}\right)+\frac{\chi}{\phi}=0 \tag{4}$$

which is the differential equation. Making the substitution

$$u=\frac{\psi}{\phi}, \qquad v=\frac{\chi}{\phi} \tag{5}$$

equation (4) takes the form

$$\left(\frac{dv}{du}\right)^2-u\frac{dv}{du}+v=0$$

which is clearly in Clairaut's form

$$v=uP-P^2 \cdots\cdots \left(P=\frac{dv}{du}\right). \tag{6}$$

Now
$$P=\frac{u\pm\sqrt{u^2-4v}}{2},$$

and setting $u^2-4v=w^2$, we find $dw=\pm du$, giving $w=\pm u-2c$. Hence, for both signs,

$$v=cu-c^2 \tag{7}$$

showing that $P=c$ gives the general solution of Clairaut's equation (6). On differentiating (6), we find

$$(u-2P)\frac{dP}{du}=0. \quad \text{Setting} \frac{dP}{du}=0$$

we find $P=C$, as already shown. From $u-2P=0$, we see that equation (6) has another integral, the curve being expressed in parametric form

$$v=uP-P^2, \quad u=2P.$$

Eliminating P between these two equations, we find exactly the envelope of equation (7), namely

$$u^2-4v=0. \tag{8}$$

3. These relationships, however, do not aid us in solving the direct problem, when the differential equation is written in the usual form

$$F(x,y)p^2+G(x,y)p+H(x,y)=0 \tag{9}$$

where
$$p=\frac{dy}{dx},$$

and F, G, H are rational integral functions of the second degree in x and y. If equation (9) could readily be expressed in the form (4),

the solution would automatically follow, in view of having obtained the transformation (5); and it would not be necessary, as shown above, to express equations (5) in the form

$$x=X(u,v), \quad y=Y(u,v) \tag{10}$$

which in the general case would be complicated expressions involving radicals, and make these substitutions (10) in equation (9).

4. A method of solving the general case is suggested by the consideration that the envelope appears in both the p- and c-discriminants. Accordingly, from equations (2) and (9), we have

$$G^2(x,y)-F(x,y)\cdot H(x,y)=\{\psi^2(x,y)-\phi(x,y)\cdot\chi(x,y)\}\Theta(x,y) \tag{11}$$

where $\Theta(x,y)$ is some expression not contained in the envelope. Should $\Theta(x,y)$ consist of three squared linear factors in x and y, these lines would technically represent tac-loci, but may satisfy the differential equation. In that case, if the envelope is a complete quadrilateral, $\Theta(x,y)=0$ would be the three diagonals, viz., the three degenerate cases of the one parameter family of conics touching the four sides of the quadrilateral.

The general solution can now be found, not by integrating the differential equation, but by determining the one-parameter family of conics touching the four straight lines of the envelope, with which the general solution is identical. Thus the general solution is obtained by resort to the methods of analytic geometry.

5. Before attacking the general case, it may prove instructive to give some simple examples falling into different categories. The case to be considered first is the differential equation (9) in which the first degree term in p is missing, viz., $H(x,y)=0$. This is a special case of Euler's equation

$$\text{(a)} \quad \frac{dx}{\sqrt{X}} \pm \frac{dy}{\sqrt{Y}}=0 \tag{12}$$

where

$$X=a_0x^4+a_1x^3+a_2x^2+a_3x+a_4,$$

$$Y=a_0y^4+a_1y^3+a_2y^2+a_3y+a_4,$$

and these expressions are not perfect squares.

Setting $\quad a_0=a_1=0, \quad h=\dfrac{-a_3}{2a_2}, \quad a_4=\dfrac{a_3^2-4a_2^2}{4a_2},$

and making the substitution $x=\xi+h$, $y=\eta+h$, equation (12) reduces to

$$\frac{d\xi}{\sqrt{a_2(\xi^2-1)}} \pm \frac{d\eta}{\sqrt{a_2(\eta^2-1)}}=0. \tag{13}$$

Writing this as an equation in x and y, for convenience, in the form

$$\frac{dx}{\sqrt{1-x^2}} \pm \frac{dy}{\sqrt{1-y^2}} = 0 \tag{14}$$

the solution by inspection is

$$\sin^{-1}x \pm \sin^{-1}y = c. \tag*{(14)$_a$}$$

Setting $\sin^{-1}x = \theta$, $\sin^{-1}y = \phi$, we have

$$\theta \pm \phi = \sin^{-1}c_1 = \cos^{-1}c_2 = \tan^{-1}c_3 = 2\cot^{-1}c_4, \text{ say.}$$

Hence four other forms of the solution are

$$x\sqrt{1-y^2} \pm y\sqrt{1-x^2} = c_1 \tag*{(14)$_b$}$$

$$\sqrt{1-x^2}\cdot\sqrt{1-y^2} \pm xy = c_2 \tag*{(14)$_c$}$$

$$x\sqrt{1-y^2} \pm y\sqrt{1-x^2} = c_3(\sqrt{1-x^2}\cdot\sqrt{1-y^2} \mp xy) \tag*{(14)$_d$}$$

$$\sqrt{1-x^2} \pm \sqrt{1-y^2} = c_4(x \mp y) \tag*{(14)$_e$}$$

Rationalizing the form $(14)_c$, after transposing $\pm xy$ to the right hand side of the equation, we readily find for the general solution

$$c^2 + (2xy)c + x^2 + y^2 - 1 = 0 \quad (\pm c_2 = c). \tag{15}$$

From equation (14) written in the form

$$(1-x^2)p^2 - (1-y^2) = 0 \tag*{(14)$'$}$$

and equation (15), the p- and c-discriminants are found to be identical, giving the envelope

$$(1-x^2)(1-y^2) = 0 \tag{16}$$

which represents four straight lines forming a square

$$x-1=0, \quad x+1=0, \quad y-1=0, \quad y+1=0. \tag*{(16)$'$}$$

Equation (15) represents the one-parameter family of conics touching the sides of this square.* Note that, starting with the general solution (15), the differential equation is found to be $(y^2-x^2)(x^2-1)p^2 - (y^2-x^2)(1-y^2)=0$. Using this form, the p-discriminant is $(x+y)^2(y-x)^2(1-x^2)(1-y^2)=0$. The equations $(y+x)^2=0$ and $(y-x)^2=0$ are the values taken by (15) for $c=+1$ and $c=-1$, respectively. They satisfy the differential equation.

*For another treatment of this same case of Euler's equation, consult Edouard Goursat, *A Course in Mathematical Analysis: Differential Equations*, being Part II of Volume II. Translation into English by E. R. Hedrick and O. Dunkel (Ginn and Co., 1917), pp. 23-24.

(b) Setting $x=\xi/a$, $y=\eta/b$ in equations (14)′ and (15), we see at once that the general solution of the differential equation

$$(a^2-\xi^2)P^2-(b^2-\eta^2)=0 \left(P=\frac{dy}{d\xi}\right) \tag{17}$$

is

$$c^2+\left(\frac{2\xi\eta}{ab}\right)c+\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}-1=0 \tag{18}$$

giving the envelope consisting of four straight lines

$$\xi-a=0 \quad \xi+a=0, \quad \eta-b=0, \quad \eta+b=0 \tag{19}$$

forming a rectangle.*

(c) Making the transformation

$$x=\frac{1}{\sqrt{2}}(X-Y), \quad y=\frac{1}{\sqrt{2}}(X+Y),$$

constituting a counter-clockwise rotation of the axes through 45°, we find from equations (14)′ and (15), that the general solution of the differential equation

$$XYP^2-(X^2+Y^2-2)P+XY=0 \quad (P=dY/dX) \tag{20}$$

is

$$c^2+(X^2-Y^2)c+(X^2+Y^2-1)=0 \tag{21}$$

giving the envelope consisting of the four straight lines

$$\begin{aligned} &X+Y-\sqrt{2}=0, \quad X+Y+\sqrt{2}=0,\\ &X-Y-\sqrt{2}=0, \quad X-Y+\sqrt{2}=0 \end{aligned} \tag{22}$$

forming a square. The general solution (21) may be written

$$Y^2C^2-(X^2+Y^2-2)C+X^2=0 \left(c=\frac{1+C}{1-C}\right) \tag{21$'$}$$

or

$$X^2c_1^2-(X^2+Y^2-2)c_1+Y^2=0 \quad (C=1/c_1). \tag{21$''$}$$

It is interesting to note that setting $X^2=u$, $Y^2=v$ in (20), we have Clairaut's form

$$v=up_1-\frac{2p_1}{1-p_1} \left(p_1=\frac{dv}{du}\right) \tag{20$'$}$$

*Joseph Edwards, *An Elementary Treatise on the Differential Calculus* (2nd edition: Macmillan, 1892), Ex. 11, p. 296.

giving the general solution

$$Y^2 = c_1 X^2 - \frac{2c_1}{1-c_1}$$

which, on clearing out, is identical with (21)″.* If we set $X^2 - Y^2 = u$, $X^2 + Y^2 - 1 = v$ in (20), we reach Clairaut's form

$$v = up_2 - p_2^2 \text{ where } p_2 = \frac{dv}{du} \tag{22}$$

giving the general solution

$$c_2^2 - (X^2 - Y^2)c_2 + (X^2 + Y^2 - 1) = 0 \tag{23}$$

which is identical with (21), for $c_2 = -c$. For the substitution $X^2 - Y^2 = u$, $X^2 + Y^2 = v$, equation (20) takes Clairaut's form

$$u = vp_3 + \frac{1-p_3^2}{p_3}, \quad \text{where } p_3 = \frac{dv}{du}$$

giving the general solution, for $p_3 = c_3$

$$(X^2 + Y^2 - 1)c_3^2 - (X^2 - Y^2)c_3 + 1 = 0$$

which is identical with (21), for $c_3 = -1/c$.

6. (a) Another type of Euler's equation is

$$\frac{dx}{\sqrt{x(1-ax)}} \pm \frac{dy}{\sqrt{y(1-ay)}} = 0. \tag{24}$$

The solution, as in article 5, using the formula

$$\int \frac{du}{(u-a)(b-u)} = 2 \sin^{-1} \sqrt{\frac{u-a}{b-a}}$$

is

$$\sqrt{1-ax} \cdot \sqrt{1-ay} - \sqrt{ax} \cdot \sqrt{ay} = C. \tag{25}$$

Rationalizing we have, after adding and subtracting $4xy$, and setting

$$c = \frac{1-C^2}{a},$$

$$c^2 - 2(x+y-2axy)c + (x-y)^2 = 0. \tag{25'}$$

*This is problem 11, p. 49, in D. A. Murray, *Introductory Course in Differential Equations* (Longmans, Green and Co., all editions).

The p- and c-discriminants are identical with the envelope

$$xy(1-ax)(1-ay)=0 \tag{26}$$

giving the four straight lines

$$x=0, \quad y=0, \quad ax-1=0, \quad ay-1=0$$

(b) We may readily set up the differential equation having for envelope the rectangle formed by the four straight lines $x=A+a$, $x=A-a$, $y=B+b$, $y=B-b$. Setting $\xi=x-A$, $\eta=y-B$ in equation (17), we have the differential equation

$$\{a^2-(x-A)^2\} \cdot p^2-\{b^2-(y-B)^2\}=0 \left(p=\frac{dy}{dx} \right) \tag{27}$$

which has the general solution

$$c^2+\frac{2(x-A)\,(y-B)}{ab}c+\frac{(x-A)^2}{a^2}+\frac{(y-B)^2}{b^2}-1=0 \tag{28}$$

and the envelope

$$1-\frac{(x-A)^2}{a^2}-\frac{(y-B)^2}{b^2}+\frac{(x-A)^2(y-B)^2}{a^2b^2}=0 \quad \text{or} \tag{29}$$

$$\left\{1-\frac{(x-A)^2}{a^2}\right\}\left\{1-\frac{(y-B)^2}{b^2}\right\}=0 \tag{29$'$}$$

giving the rectangle having as sides the four straight lines

$$x=A+a, \quad x=A-a, \quad y=B+b, \quad y=B-b.$$

7. Let us consider next the differential equation

$$(m^2xy)p^2+(n^2-l^2x^2-m^2y^2)p+l^2xy=0 \tag{30}$$

of which a special case ($l=m=1$, $n=\sqrt{2}$), equation (20), was discussed in article 5. By means of the substitution $x^2=u$, $y^2=v$, equation (30) reduces to Clairaut's form

$$v=uP+\frac{n^2P}{m^2P-l^2} \cdots \left(P=\frac{dv}{du} \right) \tag{31}$$

giving the general solution

$$y^2=x^2c+\frac{n^2c}{m^2c-l^2} \quad \text{or} \tag{32}$$

$$(m^2x^2)c^2-(l^2x^2+m^2y^2-n^2)c+l^2y^2=0 \tag{32$'$}$$

and the envelope

$$(33)\qquad (l^2x^2+m^2y^2-n^2)^2-4l^2m^2x^2y^2=0 \qquad \text{or}$$

$$(33)'\qquad (lx\pm my\pm n)^2=0.$$

These are four lines forming a rhombus (Fig. 1).

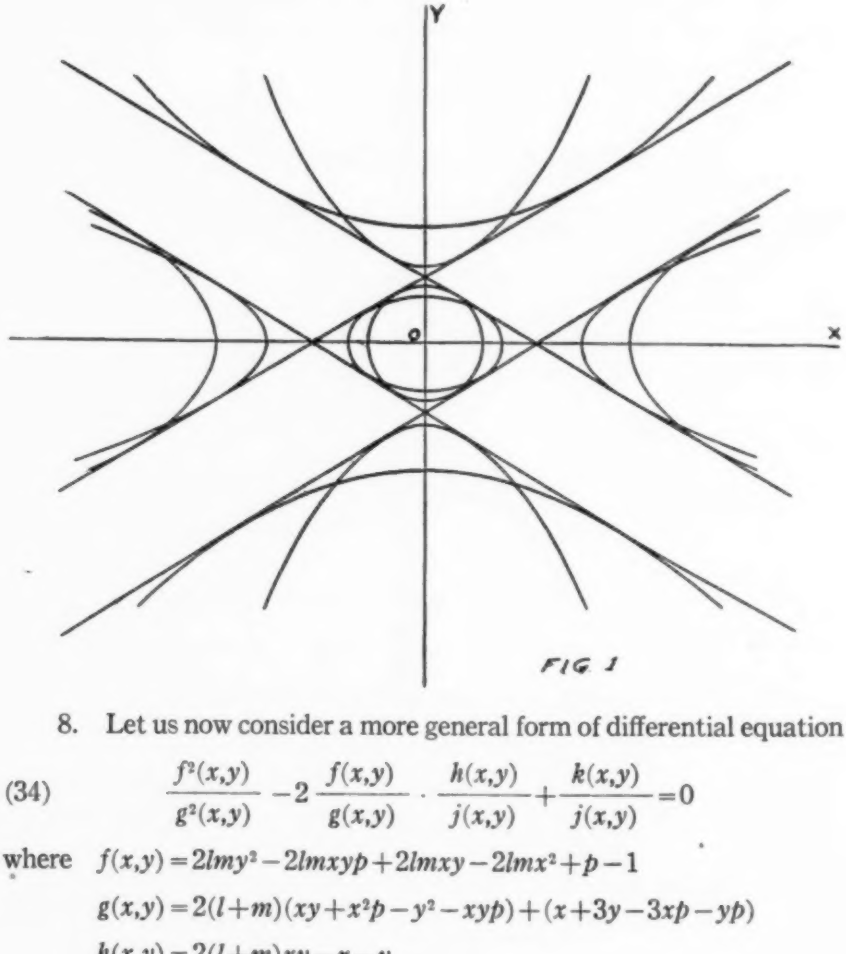

FIG 1

8. Let us now consider a more general form of differential equation

$$(34)\qquad \frac{f^2(x,y)}{g^2(x,y)}-2\frac{f(x,y)}{g(x,y)}\cdot\frac{h(x,y)}{j(x,y)}+\frac{k(x,y)}{j(x,y)}=0$$

where $f(x,y)=2lmy^2-2lmxyp+2lmxy-2lmx^2+p-1$

$g(x,y)=2(l+m)(xy+x^2p-y^2-xyp)+(x+3y-3xp-yp)$

$h(x,y)=2(l+m)xy-x-y$

$j(x,y)=(x-y)^2$

$k(x,y)=1-4lmxy.$

This equation, when cleared out and before association and cancellation of like terms, contains 2,619 terms; and reduces to the usual

form of equation (9), article 3. The p-discriminant, after laborious calculations, turns out to be

$$(x-y)^2(lx+ly-1)^2(mx+my-1)^2 \tag{35}$$

$$xy(lx+my-1)(mx+ly-1)=0$$

Since the equations $x-y=0$, $lx+ly-1=0$, $mx+my-1=0$ apparently constitute the tac-locus, but satisfy equation (34), we infer that the envelope is

$$xy(lx+my-1)(mx+ly-1)=0. \tag{36}$$

Such a differential equation as (34), with its hundreds of terms, defies direct solution because of the lack of any clues as to the substitution which will lead to Clairaut's form. Consequently it becomes necessary to discover the general solution by determining the one-parameter family of conics which touches the four lines (36); and this will be the general solution of (34).

The equation of the family of conics touching the axes of x and y is

$$(Ax+By-1)^2-2\lambda xy=0 \tag{37}$$

where certain restrictions may be placed upon λ in order that the conic touch the two lines $lx+my-1=0$, $mx+ly-1=0$.* The lines joining the origin to the points where the conic (37) cuts the line $lx+my-1=0$ are given by the equation

$$(Ax+By-lx-my)^2-2\lambda xy=0. \tag{38}$$

Applying the condition for tangency, namely that (38) be a perfect square, we find

$$\lambda=2(A-l)(B-m) \tag{39}$$

Similarly the condition for tangency for the line $mx+ly-1=0$ is

$$\lambda=2(A-m)(B-l). \tag{40}$$

From (39) and (40) we have $A=B$. Therefore the one-parameter family of conics touching the four lines (36) is

$$(x-y)^2c^2+2\{2(l+m)xy-x-y\}c+1-4lmxy=0 \tag{41}$$

where $c=A$, and this is the general solution of the differential equation (34). The c-discriminant is then

$$\{2(l+m)xy-x-y\}^2-(x-y)^2(1-4lmxy)=0$$

*C. Smith, *Conic Sections* (Macmillan, 1912, new edition), pp. 298-299.

which reduces to the form (36) already found for the envelope. From article 2 it is obvious, after the general solution (36) has been found, that the substitution

$$(42) \qquad \frac{4(l+m)xy-2(x+y)}{(x-y)^2}=u, \qquad \frac{1-4lmxy}{(x-y)^2}=v.$$

will reduce the differential equation (34) to Clairaut's form

$$v=\pm uP-P^2$$

giving the solution

$$\frac{1-4lmxy}{(x-y)^2}=\pm\frac{4(l+m)xy-2(x+y)}{(x-y)^2}c-c^2$$

which immediately reduces to the general solution (41). Letting

$$c=\frac{1}{C}$$

in (41), the resulting equation, for the values

$$C=0, \quad \frac{1}{l}, \quad \frac{1}{m}$$

respectively, takes the forms

$$(x-y)^2=0, \quad (lx+ly-1)^2=0, \quad (mx+my-1)^2=0;$$

These technically represent the tac-locus, but they satisfy the differential equation (34). Hence they must constitute the three diagonals of the complete quadrilateral, being degenerate forms of the one-parameter family of conics (41). (See Fig. 2).

9. The task of solving the general case, where the quadrilateral is composed of any four lines

$$a_ix+b_iy+c_i=0 \quad (i=1,2,3,4)$$

is laborious in the extreme. The differential equation chosen for solution in the present article has as envelope a quadrilateral composed of four lines, in which a_i, b_i, c_i, are given arbitrary numerical values; and contains, before association and cancellation of like terms, the impressive total of 63,482 terms.* The equation chosen for our study is of form (34), here given the number (42), where

*This equation has been computed from data in a thesis for the M. A. degree, carried out under the writer's direction, dealing with the converse of the problem treated in the present paper: *Methods of Finding the Differential Equation whose Envelope is Four Straight Lines*, by Louise Adams, University of North Carolina, 1930.

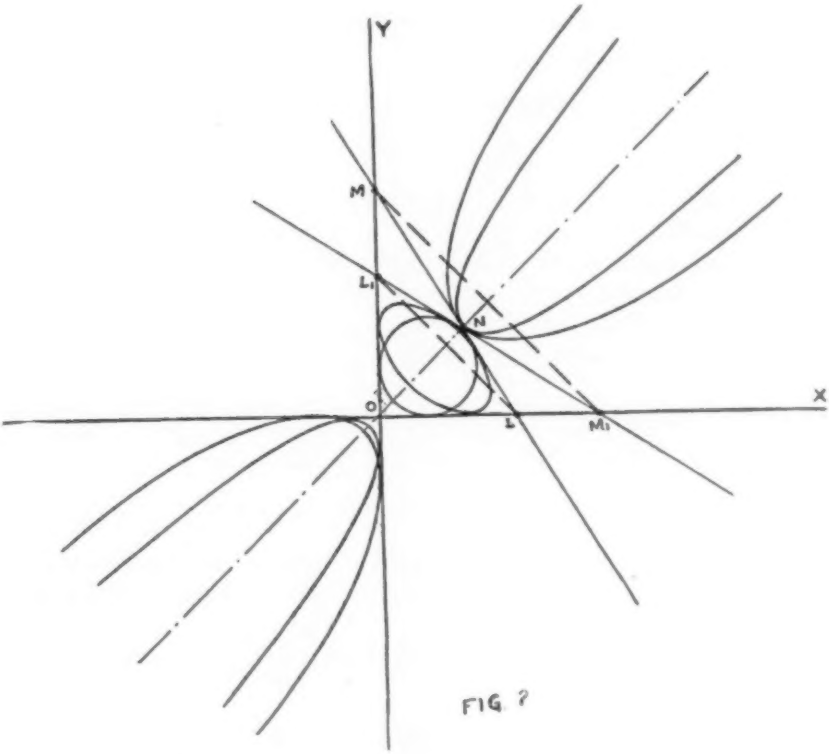

FIG 8

$$f(x,y) = -3{,}768{,}750xy^2+3{,}150{,}000x^2y-900{,}000x^2+5{,}850{,}000xy$$
$$-1{,}620{,}000x+3{,}769{,}000x^2yp+5{,}062{,}500x^2yp+262{,}500xyp$$
$$+4{,}172{,}500y^2p-2{,}711{,}250yp-4{,}550{,}000x^3p-5{,}625{,}000x^2p$$
$$-2{,}160{,}000xp-5{,}062{,}500y^3+731{,}250y^2+2{,}340{,}000y$$
$$-720{,}000+315{,}000p.$$

$$g(x,y) = -6{,}840{,}000xy^2+2{,}520{,}000x^2y+864{,}000x^2+4{,}464{,}000xy$$
$$+2{,}278{,}400x+6{,}840{,}000x^2yp+127{,}800y^2p+7{,}146{,}000xyp$$
$$-792{,}300y^2p+1{,}486{,}800yp+448{,}000x^3p-3{,}600{,}000x^2p$$
$$-554{,}340xp-5{,}006{,}250y^3-5{,}544{,}000y^2+1{,}754{,}340y$$
$$+825{,}200+525{,}660p.$$

$$h(x,y) = 800x^2+4100y^2-1850xy-200x-1850y-1000.$$

$$j(x,y) = 1120x^2+1465y^2+560xy+800x+200y+176.$$

$$k(x,y) = 625x^2+2500y^2-2500xy+1250x-2500y+625.$$

After long and laborious calculations, the p-discriminant is found to be

$$(43)\qquad (x-2y+1)^2(10x+5y+6)^2(11x-5y+15)^2(x+y+1)$$
$$(2x-y+2)(3x-2y-1)(x-3y+2)=0.$$

The equations $x-2y+1=0$, $10x+5y+6=0$, $11x-5y+15=0$ satisfy the differential equation (42) and so cannot constitute the tac-locus. They must represent degenerate forms of the one-parameter family of conics which we shall derive in the sequel, notably the three diagonals (AC,BD,EF, Fig. 3) of the complete quadrilateral formed by the four lines which constitute the envelope

$$(44)\qquad (x+y+1)(2x-y+2)(3x-2y-1)(x-3y+2)=0 \qquad \text{or}$$

$$(44)'\qquad 6x^4-19x^3y-2x^2y^2+17xy^3-6y^4+22x^3-38x^2y+7xy^2$$
$$+7y^3+22x^2-23xy+13y^2+2x-4y-4=0.$$

The next step is to find the one-parameter family of conics touching the four lines given by (44), and this will be the general solution of the differential equation (42).

Consider the general homogeneous equation of a conic in point co-ordinates

$$(45)\qquad ax^2+by^2+cz^2+2hxy+2gzx+2fyz=0.$$

This same equation in line co-ordinates is

$$(46)\qquad Au^2+Bv^2+Cw^2+2Huv+2Gwu+2Fvw=0$$

where A,B,C,F,G,H are the respective minors of the determinant

$$(47)\qquad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$$

The equations of the four lines given by (44), expressed in homogeneous co-ordinates, are

(α) $x+y+z=0$, where $(u,v,w)=(1,1,1)$

(β) $2x-y+2z=0$, where $(u,v,w)=(2,-1,2)$

(γ) $3x-2y-z=0$, where $(u,v,w)=(3,-2,-1)$

(δ) $x-3y+2z=0$, where $(u,v,w)=(1,-3,2)$.

The conic will touch the lines (α), (β), (γ), (δ) if the points $(1,1,1)$, $(2,-1,2)$, $(3,-2,-1)$, $(1,-3,2)$ respectively satisfy (46).

The conditions are

$$A+B+C+2H+2G+2F=0$$
$$4A+B+4C-4H+8G-4F=0$$
$$9A+4B+C-12H-6G+4F=0$$
$$A+9B+4C-6H+4G-12F=0.$$

Solving these four equations for A, B, C, and H in terms of G and F, we find

$$A=\frac{-504G+2100F}{-1050}, \quad B=\frac{-672G}{-1050},$$

$$C=\frac{2940G-2100F}{-1050}, \quad H=\frac{168G+1050F}{-1050}.$$

Substituting these values in equation (46) we have, after reduction,

(48) $\quad (6c-25)u^2+8cv^2-(35c-25)w^2-(4c+25)uv+25cuw+25vw=0$

where $c=G/F$.

For the conic in point co-ordinates (45), the coefficients a,b,c,h,g,f are the respective minors of the determinant

$$\begin{vmatrix} 6c-25, & -\frac{(4c+25)}{2}, & \frac{25c}{2} \\ -\frac{(4c+25}{2} & 8c & \frac{25}{2} \\ \frac{25c}{2} & \frac{25}{2} & -(35c-25) \end{vmatrix}=0$$

Hence we have $\quad a=\dfrac{-1120c^2+800c-625}{4}$

$$b=\frac{-1465+4100c-2500}{4}$$

$$c=\frac{176c^2-1000c-625}{4}$$

$$h=\frac{-280c^2-925c+1250}{4}$$

$$g = \frac{-400c^2 - 100c - 625}{4}$$

$$f = \frac{-100c^2 - 925c + 1250}{4}.$$

Substituting these values in the point equation of the conic (45) we find after reduction and replacing x/z, y/z by x, y respectively

$$(49) \qquad \begin{aligned} &(1120x^2 + 1465y^2 + 560xy + 800x + 200y - 176)c^2 \\ &-(800x^2 + 4100y^2 - 1850xy - 200x - 1850y - 1000)c \\ &+(625x^2 + 2500y^2 - 2500xy + 1250x - 2500y + 625) = 0. \end{aligned}$$

This is the general solution of the differential equation (42).

The c-discriminant takes the form, after division by 360,000, of (44)′, and thus is the anticipated envelope.

It may be observed, on the basis of article 2, that if we make in the original differential equation the substitution

$$(50) \qquad \begin{cases} u = \dfrac{800x^2 + 4100y^2 - 1850xy - 200x - 1850y - 1000}{1120x^2 + 1465y^2 + 560xy + 800x + 200y - 176}, \\[2ex] v = \dfrac{625x^2 + 2500y^2 - 2500xy + 1250x - 2500y + 625}{1120x^2 + 1465y^2 + 560xy + 800x + 200y - 176} \end{cases}$$

we immediately reduce it to Clairaut's form

$$P^2 - Pu + v = 0 \cdots \left(P = \frac{dv}{du} \right),$$

giving the general solution,

$$c^2 - cu + v = 0.$$

These results are identical with equations (42) and (49) respectively.

For special values of c, the general solution (49) takes degenerate forms of the one-parameter family of conics, notably the three diagonals of the complete quadrilateral. For example, the general solution (49), for $c = 0$, takes the form, after division by 625,

$$(x - 2y + 1)^2 = 0.$$

See Fig. 3.

10. The text-books on differential equations are conspicuous for the omission of the two familiar curves presenting type-examples

of the nodal and cuspidal locus, respectively: the folium of Descartes and the cissoid of Diocles. These examples will now be handled.

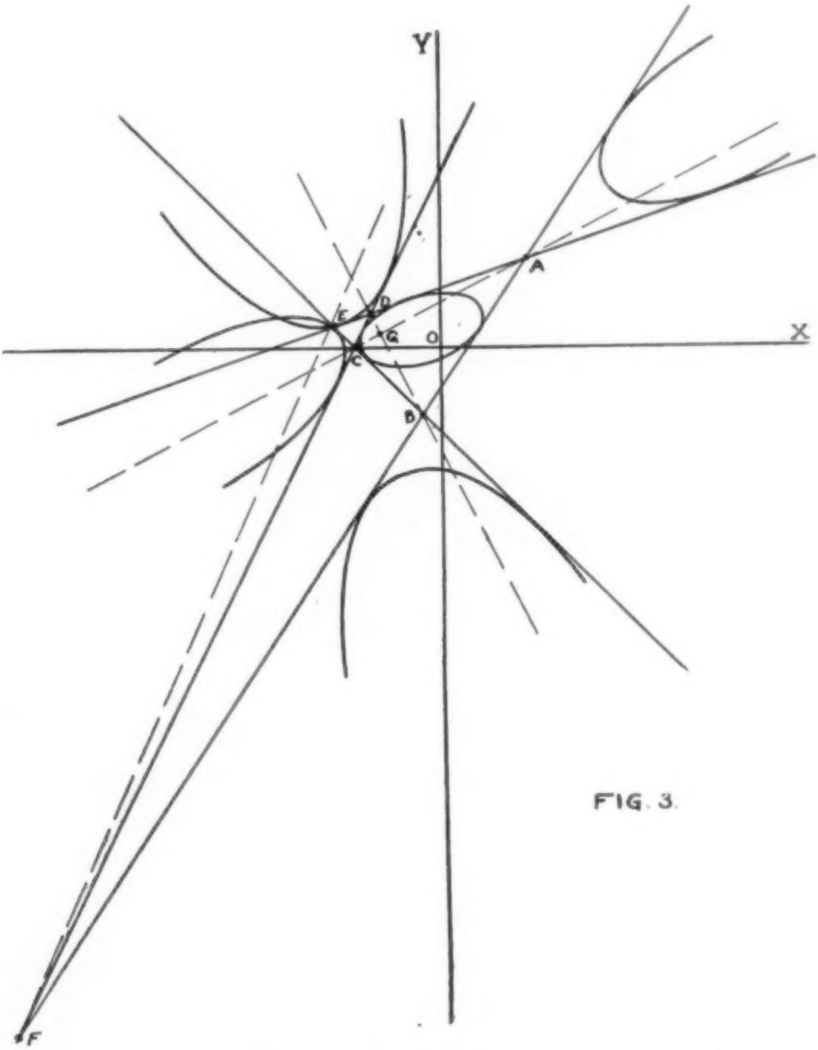

FIG. 3.

A. Given the differential equation

$$(y^4-2y)p^3+3p^2+(y^4-4y)=0. \tag{51}$$

Making the substitution

$$y=\frac{3t^2}{1+t^3}$$

in (51), we have on reduction

$$27(2+10t^3-t^6)(1-2t^3)(2-t^3)p_1^3-9(2-t^3)^2(1+t^3)^6p_1^2 \tag{52}$$
$$+(2-t^3)(2+7t^3-4t^6)(1+t^3)^6=0\cdots\left(p_1=\frac{dt}{dx}\right).$$

Dividing out by $(2-t^3$, we find one root of the resulting cubic equation to be

$$p_1=\frac{(1+t^3)^2}{3(1-2t^3)} \tag{53}$$

giving

$$dx=\frac{3(1-2t^3)}{(1+t^3)^2}\,dt=\frac{3(1+t^3)-3t(3t^2)}{(1+t^3)^2}\,dt. \tag{54}$$

Hence
$$x+c=\frac{3t}{1+t^3}\ ;$$

and eliminating t between this equation and

$$y=\frac{3t^2}{1+t^3}\ ,$$

the other equation of the parametric pair, we find for the general solution

$$(x+c)^3+y^3-3(x+c)y=0 \tag{55}$$

which is the familiar equation of the folium of Descartes. We obtain the following forms for the p- and c-discriminants, respectively

$$y(y^3-4)(y^{12}-8y^9+20y^6-16y^3+4)=0 \tag{56}$$

(57 $$y^3(y^3-4)=0.$$

Hence we have the following results

$$y(y^3-4)=0\cdots\cdot\text{Envelope} \tag{58}$$

$$y^2=0\cdots\text{Nodal locus} \tag{59}$$

$$\left.\begin{cases} y^{12}-8y^9+20y^6-16y^3+4=0 \\ \text{or}\quad (y^6-4y^3+2)^2=0 \\ \text{or}\quad \{(y^3-2)^2-2\}^2=0 \\ \text{or}\quad y=\sqrt[3]{2\pm\sqrt{2}} \end{cases}\right\}\cdots\text{Tac Locus.} \tag{60}$$

The features of the configuration are indicated in the diagram. See Fig. 4.

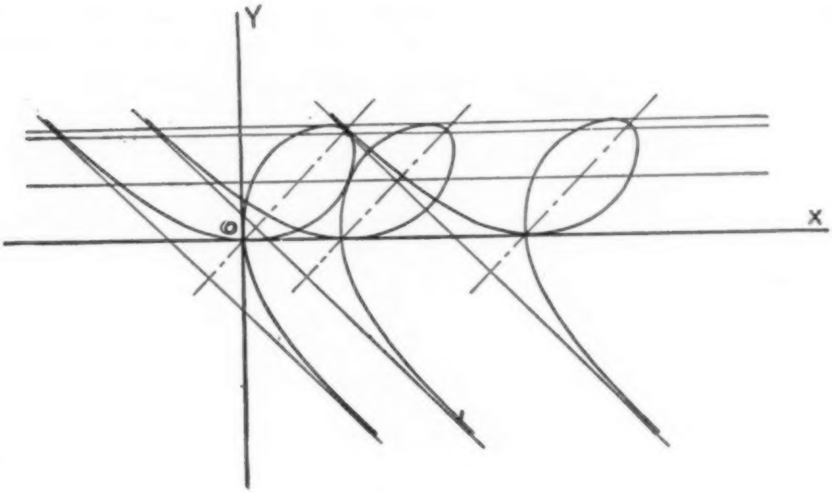

FIG. 4

B. Next, let us consider the differential equation

$$(2a-x)^3p^2-x(3a-x)^2=0. \tag{61}$$

Making the transformation $x=\dfrac{2au^{2}*}{1+u^2}$, we find

$$dy=\frac{4au^2}{(1+u^2)^2}\cdot du+2a\frac{u^2}{1+u^2}\cdot du \tag{62}$$

giving the general solution

$$y+c=\frac{2au^3}{1+u^2}=\pm\frac{x^{3/2}}{(2a-x)^{1/2}}.$$

On rationalizing, we recognize this as the equation of the cissoid of Diocles in the familiar form

$$(y+c)^2=\frac{x^3}{(2a-x)}. \tag{64}$$

From the p- and the c-discriminants, respectively

$$x(2a-x)^3(3a-x)^2=0 \tag{65}$$

$$x^3(2a-x)^3=0 \tag{66}$$

*An alternative substitution is $x=2a\sin^2\theta$.

we see that $x=0$ is both envelope and cuspidal locus; and that $x=2a$ is also a part of the envelope. The equation $x=3a$ is an extraneous solution, having no geometric relation to the problem. See Fig. 5.

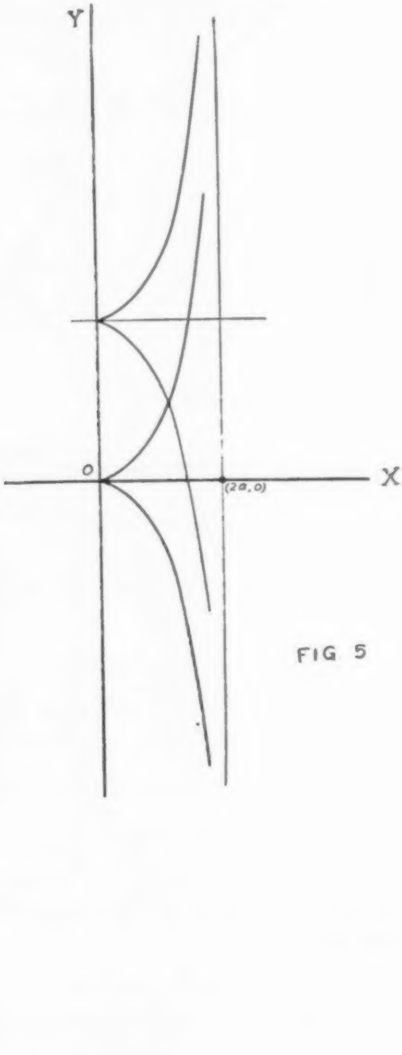

FIG 5

Integration of Functions in a Banach Space

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1. In a paper by R. L. Jeffery [1], an integral was defined which is equivalent to the well-known Birkhoff integral [2], but simpler in that it requires no preliminary study of convex sets and convergence in abstract space. In the present paper the reduction is carried further, and an integral equivalent to Birkhoff's is given in what seems to be the simplest possible form. Then, following the lines indicated by G. B. Price [3], the notion of measure-function used in defining the integral is enlarged.

As in the three papers mentioned, we consider a function $f(x)$ from a space $\mathbf{X}$ with elements x, on which a measure is defined, to a Banach space $\mathfrak{B}$. More precisely, let there be a family $\mathcal{A}$ of sets A in $\mathbf{X}$, such that $\mathcal{A}$ contains the null-set, the complement of any set A belonging to $\mathcal{A}$, and the sum of any sequence of sets A_n belonging to $\mathcal{A}$. Let there be defined a completely additive, numerically valued measure-function $m(A)$ on $\mathcal{A}$. All sets in $\mathbf{X}$ hereafter mentioned will be understood to belong to $\mathcal{A}$. The Banach space $\mathfrak{B}$ is, according to the standard definition, a normed, complete vector space.

We begin by considering bounded functions on a set P with $m(P)<\infty$, and take up the unbounded functions afterwards. By the same method we might also include the case $m(P)=\infty$, but throughout the paper we shall retain for simplicity the assumption that $m(P)<\infty$.

2. *Integral of a bounded function.* Throughout this section we shall assume that $f(x)$ is bounded on a set P in $\mathbf{X}$. Let $\mathcal{P}$ be a finite partition of P; that is, let $\mathcal{P}$ denote a family of non-overlapping sets p_i $(i=1, 2,\cdots, k)$ with $\Sigma p_i=P$. Define

$$S(\mathcal{P})=\Sigma_i m(p_i)f(x_i),$$

$$D(\mathcal{P})=\Sigma_i m(p_i)[f(x_i')-f(x_i'')],$$

where x_i, x_i', x_i'', are points of p_i. Also let

$$\omega(\mathcal{P})=\sup\|D(\mathcal{P})\|,$$

as x_i', x_i'' vary. Thus $S(\mathcal{P})$ and $D(\mathcal{P})$ depend on the choice of x_i, x_i', x_i'', although this is not indicated explicitly in the symbol, but

$\omega(\mathcal{P})$ depends on $\mathcal{P}$ alone. If other functions besides $f(x)$ are involved, we shall sometimes write $S(f,\mathcal{P})$ and so on. Whenever a sum such as $\Sigma_i m(p_i)f(x_i)$ is written, it will be understood that x_i is a point on p_i. By the *product* $\mathcal{P}_1 * \mathcal{P}_2$ of two partitions we mean the partition consisting of all sets $p_i^{(1)}p_j^{(2)}$.

Definition. *If there is a sequence of partitions* $\{\mathcal{P}_n\}$ *of such a nature that* $S(\mathcal{P}_n)$ *tends to a limit* $K=K(f,P)$ *which does not depend on the choice of* $\{x_{ni}\}$ *in each partition, we shall say that* $f(x)$ *is integrable, and shall call* K *the integral of* $f(x)$ *over* P.

In connection with this definition we shall prove the following theorems, which are of fundamental importance under the present approach to the subject.

Theorem 1. *A necessary and sufficient condition for* $S(\mathcal{P}_n)$ *to have a limit which does not depend on* $\{x_{ni}\}$, *is that*

$$\omega(\mathcal{P}_n)\to 0. \tag{1}$$

Theorem 2. (Uniqueness theorem). *If* $S(\mathcal{P}_n)$ *tends to a limit* K *which does not depend on* $\{x_{ni}\}$, *and if there is a second sequence of partitions* $\{P'\}$ *for which* $S(\mathcal{P}'_n)$ *tends to a limit* K' *which does not depend on* $\{x'_{ni}\}$, *then* $K'=K$.

In the remainder of the paper, all statements regarding $S(\mathcal{P})$ and $D(\mathcal{P})$ for any partition, will be understood to be true for every choice of the points involved. The proofs of the above theorems depend on the following lemma.

Lemma. *Suppose a partition* $\mathcal{Q}$ *is obtained from* $\mathcal{P}$ *by subdividing each set* p_i *into a finite number of sets* q_{ij_i}. *Then*

$$\|D(\mathcal{Q})\| \leqq \omega(\mathcal{P})$$

and

$$\|S(\mathcal{P})-S(\mathcal{Q})\| \leqq \omega(\mathcal{P}).$$

Proof. We have

$$D(\mathcal{Q})=\sum_i \sum_{j_i} m(q_{ij_i})[f(x'_{ij_i})-f(x''_{ij_i})]$$

$$=\sum_i \sum_{j_i} [m(q_{ij_i})/m(p_i)]m(p_i)[f(x'_{ij_i})-f(x''_{ij_i})],$$

where, if for any set p_i we have $m(p_i)=0$, the corresponding terms are omitted from the summation. Then, putting

$$c_{ij_i}=m(q_{ij_i})/m(p_i), \quad p_{j_1j_2\cdots j_k}=c_{1j_1}c_{2j_2}\cdots c_{kj_k},$$

we have $$D(\mathcal{Q})=\sum_i \sum_{j_i} c_{ij_i}m(p_i)[f(x'_{ij_i})-f(x''_{ij_i})]$$

$$=\sum_i \sum_{j_1} \sum_{j_2}\cdots \sum_{j_k} p_{j_1j_2\cdots j_k}m(p_i)[f(x'_{ij_i})-f(x''_{ij_i})].$$

Hence, $$\|D(\mathcal{Q})\| \leqq \omega(\mathcal{P}) \sum_{j_1} \sum_{j_2} \cdots \sum_{j_k} p_{j_1 j_2 \cdots j_k}$$
$$= \omega(\mathcal{P}).$$

This is the first part of the lemma. The second part may be proved by replacing x'_{ij_i} by x_i throughout. The proof of the lemma is now complete.

Proof of Theorem 1. The condition stated is obviously necessary. To prove that it is sufficient, we shall show that $\|S(\mathcal{P}_{n_1}) - S(\mathcal{P}_{n_2})\| \leqq \omega(\mathcal{P}_{n_1}) + \omega(\mathcal{P}_{n_2})$. It will then follow that $\{S(\mathcal{P}_n)\}$ is a Cauchy sequence, and so K exists in virtue of the completeness of $\mathfrak{B}$. Let $\mathcal{Q} = \mathcal{P}_{n_1} * \mathcal{P}_{n_2}$. We have by the lemma,

$$\|S(\mathcal{P}_{n_1}) - S(\mathcal{Q})\| \leqq \omega(\mathcal{P}_{n_1}),$$
$$\|S(P_{n_2}) - S(\mathcal{Q})\| \leqq \omega(\mathcal{P}_{n_2}).$$

The result now follows.

Proof of Theorem 2. Let $\mathcal{Q} = \mathcal{P}_n * \mathcal{P}_n'$. Then

$$\|S(\mathcal{P}_n) - S(\mathcal{Q})\| \leqq \omega(\mathcal{P}_n),$$
$$\|S(\mathcal{P}'_n) - S(\mathcal{Q})\| \leqq \omega(\mathcal{P}'_n).$$

Hence, $$\|S(\mathcal{P}'_n) - S(\mathcal{P}_n)\| \leqq (\mathcal{P}'_n) + \omega(\mathcal{P}_n)$$
$$\to 0,$$

and the result follows.

This completes the proof of Theorems 1 and 2. In the following theorem are collected a number of remarks, all of which are obvious for bounded, integrable functions.

THEOREM 3. (a) *The partitions $\mathcal{P}_n$ need not belong to any particular sequence $\{\mathcal{P}_n\}$. It is sufficient if, given $\epsilon > 0$, we can find $\mathcal{P}$ with $\omega(\mathcal{P}) \leqq \epsilon$.*

(b) *If we have any sequence $\{\mathcal{P}_n\}$ which yields an integral K, any sequence obtained by subdividing the partitions $\mathcal{P}_n$ will also serve our purpose. Hence we may if we choose assume that $\mathcal{P}_n$ is a subdivision of $\mathcal{P}_{n-1}$.*

(c) *For any partition $\mathcal{P}$ we have $\|K(f,P) - S(\mathcal{P})\| \leqq \omega(\mathcal{P})$.*

(d) *If $f(x)$ is integrable over P it is integrable over any set $Q \subset P$.*

(e) *If Q consists of a selection $\mathcal{Q}$ of the sets of a partition $\mathcal{P}$, we have $\|K(f,Q) - S(\mathcal{Q})\| \leqq \omega(\mathcal{Q}) \leqq \omega(\mathcal{P})$.*

(f) *If $Q + R = P$, $QR = 0$, we have $K(f,Q) + K(f,R) = K(f,P)$.*

(g) *If $M = \sup_P \|f(x)\|$, then $\|K(f,P)\| \leqq Mm(P)$; if $m(Q) \to 0$ we have $\|K(f,Q)\| \to 0$; if $P = \Sigma_1^\infty Q_n$ (the Q_n non-overlapping), then $K(f,P) = \Sigma_1^\infty K(f,Q_n)$.*

3. *Integral of an unbounded function.* In case $f(x)$ is not bounded over P, we use a limiting process. We shall denote by $B, B_1, b_1, \cdots$, sets in P over which $f(x)$ is bounded. We make the assumption that $K(f,B)$ exists for every such B, and that $K(f,B_n)$ has the same limit, which we shall denote by $K(f,P)$, for all sequences $\{B_n\}$ such that $m(B_n)\to m(P)$. We do not assume $B_n \supset B_{n-1}$, and we may equally well state our assumption in the form: There is a vector K with the property that given $\epsilon>0$ we can find δ such that $m(P-B)\leqq\delta$ implies $\|K-K(f,B)\|\leqq\epsilon$. If this condition holds, we say that $f(x)$ is *summable* over P.

Our fundamental theorem for summable functions is the following:

Theorem 4. *A necessary and sufficient condition for $f(x)$ to be summable over P is that given $\epsilon>0$, we can find δ such that for any set b on P,*

$$m(b)\leqq\delta \text{ implies } \|K(f,b)\|\leqq\epsilon. \tag{2}$$

Proof. The condition is easily seen to be necessary. To prove that it is sufficient, let $\{B_n\}$ be any sequence with $m(B_n)\to m(P)$. Then by Theorem 3 we have

$$\|K(f,B_{n_1})-K(f,B_{n_2})\|\leqq\|K(f,B_{n_1}-B_{n_2})\|+\|K(f,B_{n_2}-B_{n_1})\|,$$

which tends to zero by our assumption. This shows that $K(f,B_n)$ is a Cauchy sequence and it is clear that all such sequences are equivalent. The result now follows.

4. *Properties of the integral.* We now give various elementary properties of the integral; in all cases the proofs are simple, and require merely the consideration of conditions (1) and (2).

Theorem 5. (a) *If $K(f,P)$ exists and $Q\subset P$, then $K(f,Q)$ exists.*

(b) *If $m(Q)\to 0$ then $K(f,Q)\to 0$.*

(c) *If $m(Q)\to m(P)$ then $K(f,Q)\to K(f,P)$.*

(d) *If $\Sigma Q_n=P$ (the Q_n non-overlapping), then $\Sigma K(f,Q_n)=K(f,P)$.*

(e) *For any real constant c, we have $K(cf,P)=cK(f,P)$.*

The proofs are omitted.

Theorem 6. *If $g(x)$ and $h(x)$ are summable, and $f(x)=g(x)+h(x)$, then $f(x)$ is summable, and $K(f,P)=K(g,P)+K(h,P)$.*

Proof. We have to show (i) that $f(x)$ is integrable over any set B where it is bounded, and (ii) that $\|K(f,B)\|\to 0$ as $m(B)\to 0$. The difficulty is that $g(x)$ and $h(x)$ are not necessarily bounded on B. However, given ϵ, we shall find a partition $\mathcal{B}$ with $\omega(f,\mathcal{B})\leqq\epsilon$. Let $M=\sup_B\|f(x)\|$. Put $B=B'+B''$ where $m(B'')\leqq\frac{1}{4}\epsilon/M$, and both

$g(x)$ and $h(x)$ are bounded on B'. Let $\mathcal{B}'$ be a partition of B' with $\omega(g,\mathcal{B}')$ and $\omega(h,\mathcal{B}')$ both $\leqq \frac{1}{4}\epsilon$. Then $\omega(f,\mathcal{B}')$ and $\omega(f,B'')$ are both $\leqq \frac{1}{2}\epsilon$, therefore $\omega(f,\mathcal{B}) \leqq \epsilon$, as required. It is then easily seen that $K(f,B) = K(g,B) + K(h,B)$ and that $\|K(f,B)\| \to 0$ as $m(B) \to 0$. The proof is now complete.

THEOREM 7. *If $g(x)$ is summable and $h(x)$ is real-valued, bounded and Lebesgue measurable on P, then $f(x) = h(x)g(x)$ is summable on P. If $\|g\| = \sup_{Q \subset P} \|K(g,Q)\|$ and $M = \sup_P |h(x)|$, then*

$$\|K(f,P)\| \leqq 2M\|g\|.$$

Proof. Suppose first that $g(x)$ is bounded. Let $\epsilon > 0$ be given, and divide the range of $h(x)$ by points a_i with $0 < a_{i+1} - a_i < \epsilon$. Let $\mathcal{P}'$ be such that $\omega(g,P') < \epsilon$, let $\mathcal{P}''$ be the partition $\{p_i''\}$ where $p_i'' = E_x[a_i \leqq h(x) < a_{i+1}]$, and let $\mathcal{P} = \mathcal{P}' * \mathcal{P}''$. Then $\omega(g,\mathcal{P}) < \epsilon$.

For any two points x_i', x_i'' on a set p_i of $\mathcal{P}$, we may write

$$h(x_i') = a_i + \eta_i', \qquad h(x_i'') = a_i + \eta_i'',$$

where $0 \leqq \eta_i' < \epsilon$, $0 \leqq \eta_i'' < \epsilon$. Then

$$\begin{aligned} D(f,\mathcal{P}) &= \Sigma_i\, m(p_i)[h(x_i')g(x_i') - h(x_i'')g(x_i'')] \\ &= \Sigma_i\, m(p_i)[a_i\{g(x_i') - g(x_i'')\} + \eta_i' g(x_i') - \eta_i'' g(x_i'')] \\ &= \Sigma_1 + \Sigma_2 - \Sigma_3, \end{aligned}$$

let us say.

Consider now the sum

$$s = c_1u_1 + c_2u_2 + \cdots + c_nu_n,$$

where the u_i are vectors and the c_i are positive constants, monotonically decreasing. We have by Abel's lemma,

$$\|s\| \leqq c_1 U,$$

where U is the greatest of the quantities

$$\|u_1\|, \quad \|u_1 + u_2\|, \cdots, \quad \|u_1 + u_2 + \cdots + u_n\|.$$

If the c_i are not all positive nor monotonically decreasing, we may rearrange them and take the positive and negative c_i separately, thus obtaining

$$\|s\| \leqq 2cU',$$

where $c = \sup|c_i|$, and $U' = \sup\|\Sigma_i u_{k_i}\|$, the sum being taken over any selection of the u_i.

We have then, if $\mathcal{Q}$ is any selection of the sets of $\mathcal{P}$,

$$\|\Sigma_1\| \leqq 2M \sup\|D(g,\mathcal{Q})\| \leqq 2M\omega(g,\mathcal{P}) \leqq 2M\epsilon.$$

Similarly, by Theorem 3(c),

$$\|\Sigma_2\| \leqq 2\epsilon \sup \|S(g,\mathcal{Q})\| \leqq 2\epsilon(\|g\|+\epsilon),$$

and the same holds for $\|\Sigma_3\|$. It now follows that

$$\|D(f,\mathcal{P})\| \leqq 2\epsilon[M+2(\|g\|+\epsilon)],$$

and so $f(x)$ is summable over P.

Next, we have

$$\|S(f,\mathcal{P})\| \leqq 2M \sup \|S(g,\mathcal{Q})\| \leqq 2M(\|g\|+\epsilon),$$

whence,

$$\|K(f,P)\| \leqq 2M\|g\|,$$

which proves the theorem in the case where $g(x)$ is bounded.

If $g(x)$ is unbounded, it is convenient to put $H(x)=h(x)+C$, choosing the constant C so that $H(x)$ is bounded away from zero. Then the function $F(x)=H(x)h(x)$ is bounded on any set where $g(x)$ is bounded, and conversely. The rest of the proof presents no difficulty, and is omitted.

These theorems will serve to indicate how the properties of the integral, such as are set forth in [2], for example, may be established under the new approach. The following theorem on additive set-functions is also of interest in the present connection. It indicates that if a function fails to be summable over P according to our definition, its integral is an unbounded set-function over the sets where it exists.

Theorem 8. *Let there be sequences B_n, B'_n such that $m(B_n)\to m(P)$, $m(B'_n)\to m(P)$. Let there be a function $K(B)$ which is completely additive over any finite collection of sets B_n, B'_n. If $K(B_n)\to a$, $K(B'_n)\to a'\neq a$, we can find a sequence B''_n such that $\|K(B''_n)\|\to\infty$.*

Proof. Put $\delta=a'-a$, $\epsilon=\|\delta\|/8$. Choose n_1 so that

$$\|K(B_{n_1})-a\|<\epsilon.$$

Then choose m_1 so that

$$\|K(B_{n_1}B'_{m_1})-KB_{n_1})\|<\epsilon$$

and

$$\|a'-K(B'_{m_1})\|<\epsilon.$$

Putting $b_1=B'_{m_1}-B_{n_1}$, we have from the above inequalities,

$$\|\delta-K(b_1)\|<3\epsilon.$$

By properly choosing n_2, m_2, n_3, $m_3,\cdots$ we can obtain a sequence $\{b_n\}$ with $m(b_n)\to 0$ and

$$\|\delta-K(b_n)\|<3\epsilon \quad (n=1,2,3,\cdots).$$

Take a subsequence $\{b'_n\}$ of $\{b_n\}$ such that $m(b'_n) < 2^{-n}$. Now, starting with b_1', choose k_1 so large that if $b_1'' = b_1' - \Sigma_{n>k_1} b'_n$, we have

$$\|K(b_1'') - K(b_1')\| < \epsilon.$$

This gives $K(b_1'') = \delta + \rho_1, \quad \|\rho_1\| < 4\epsilon = \|\delta\|/2.$

Proceeding in this way we get a sequence of non-overlapping sets $b_1'', b_2'', \cdots$, such that

$$K(b_n'') = \delta + \rho_n, \quad \|\rho_n\| < 4\epsilon = \|\delta\|/2.$$

The theorem follows on putting $B_n'' = b_1'' + \cdots + b_n''$.

5. *Relations with other integrals.* We shall first show that by suitable specialization the present integral may be made equivalent to either the Riemann or the Lebesgue integral for real-valued functions; this adds emphasis to the well-known fact that the only essential difference between these integrals is the use of Lebesgue measure in the latter. We shall then show that the integral as it stands is equivalent to Birkhoff's integral [2], taken over a domain of finite measure.

For the Riemann and Lebesgue integrals we take both $\mathfrak{X}$ and $\mathfrak{Y}$ as the real number system, with $m(A)$ the Lebesgue measure, and P the interval (0,1). It will appear that the only difference between the Riemann and the Lebesgue integral is that in the former the sets p_i of $\mathcal{P}$ are restricted to be intervals.

For the Lebesgue integral we need only show that if $f(x)$ is bounded and $K(f,P)$ exists, then $f(x)$ is Lebesgue measurable. Suppose we have a sequence $\{\mathcal{P}_n\}$ with $\omega(\mathcal{P}_n) \to 0$. By Theorem 3(b) we may suppose that $\mathcal{P}_n$ is formed by subdividing $\mathcal{P}_{n-1}$. Let $\mathcal{P}_n = \{p_{ni}\}$, let M_{ni}, m_{ni} be the maximum and minimum respectively of $f(x)$ on p_{ni}, and form the functions

$$U_n(x) = M_{ni} \text{ for } x \text{ on } p_{ni} \ (i = 1, 2, \cdots),$$

$$L_n(x) = m_{ni} \text{ for } x \text{ on } p_{ni} \ (i = 1, 2, \cdots).$$

Then $\delta_n(x) = U_n(x) - L_n(x)$ is a positive finite valued function with $\int_P \delta_n(x)dx \to 0$. Hence $\delta_n(x) \to 0$ almost everywhere. But $L_n(x) \leqq f(x) \leqq U_n(x)$. Hence $f(x) = \lim_n L_n(x)$ almost everywhere, and is therefore Lebesgue measurable.

To see that if the $\mathcal{P}_n$ are sets of intervals we get the Riemann integral, we need only observe that in this case we have

$$\int_0^1 L_n(x)dx \leqq \underline{\int_0^1} f(x)dx \leqq \overline{\int_0^1} f(x)dx \leqq \int_0^1 U_n(x)dx.$$

Hence if
$$\int_0^1 [U_n(x) - L_n(x)]dx \to 0$$
we have
$$\underline{\int_0^1} f(x)dx = \overline{\int_0^1} f(x)dx,$$
and so $f(x)$ is Riemann integrable.

The Birkhoff integral. For bounded functions the equivalence is obvious in view of the following remarks. If $f(x)$ is bounded, the infinite decompositions of Birkhoff may be replaced by finite partitions, and the diameter of $\Sigma m(p_i)f(p_i)$ which appears in [2], Theorem 13, is precisely our $\omega(\mathcal{P})$.

We now consider unbounded functions. (i) Suppose that Birkhoff's integral $J(f,P)$ exists. Then, given $\epsilon > 0$, we can find δ such that $m(b) < \delta$ implies $\|J(f,b)\| < \epsilon$. This follows from a theorem of Pettis [4], since the Birkhoff integral is a Pettis integral. It now follows easily that $K(f,P)$ exists and equals $J(f,P)$, thus proving the first part of the equivalence theorem.

(ii) Suppose now $K(f,P)$ exists. We have to find a decomposition $\mathcal{P}$ under which $\Sigma m(p_i)f(p_i)$ is unconditionally summable and has a diameter less than ϵ. Let $B_1, B_2, \cdots$ be non-overlapping sets such that $\Sigma B_j = P$ and $f(x)$ is bounded on each B_j. On each B_j take a partition $\mathcal{B}_j : \{b_{ji}\}$ with $\omega(\mathcal{B}_j) < \epsilon_j$, $\Sigma \epsilon_j = \epsilon/2$. Determine δ so that $m(b) < \delta$ implies $\|K(f,b)\| < \epsilon/2$, and then choose N so that $m(B_N + B_{B+1} + \cdots) < \delta$. Choose any finite set of the $b_{ji}(j \geq N)$, denote it by $\beta : (b_1, \cdots, b_k)$ and put $b = b_1 + \cdots + b_k$. Then, by Theorem 3(e),
$$\|K(f,b) - S(\beta)\| \leqq \omega(\beta) \leqq \Sigma_j \omega(\mathcal{B}_j) < \Sigma_j \epsilon_j = \epsilon/2.$$
But since $b \subset B_N + B_{N+1} + \cdots$, we have
$$\|K(f,b)\| < \epsilon/2.$$
Hence $\|S(\beta)\| < \epsilon$. This shows that $\Sigma m(b_{ji})f(b_{ji})$ is unconditionally summable. We have, finally
$$[\Sigma m(b_{ji})f(b_{ji})] \leqq \Sigma \omega(\mathcal{B}_j) < \Sigma \epsilon_j = \epsilon/2.$$
which in view of [2], Theorem 13, completes the proof of the equivalence of the two integrals.

For the reader who wishes to compare the present development with the others, the main points are here indicated. Notice first that the developments of Jeffery and Birkhoff are respectively analogous to two methods of developing the Lebesgue integral of a real function.

Either we take bounded functions first and then proceed to the unbounded case, or we subdivide the entire y-axis from the start. The present development is essentially that of Jeffery, but we have simplified the treatment by using finite partitions when dealing with bounded functions, and adapting the ideas of Birkhoff's paper, without explicitly mentioning the closure of the convex hull. Then, for unbounded functions the notion of unconditional convergence is replaced by the treatment in sections 3 and 4, where Jeffery's methods are systematized by the use of Theorem 4 of this paper.

6. *The Price integral.* We now make the extension suggested by G. B. Price [3]. Let $\mathfrak{T}$ be the normed linear space of linear transformations T whose domain is $\mathfrak{B}$ and whose range is in $\mathfrak{B}$. Then if $T, T_1, \cdots$ are in $\mathfrak{T}$, we define $\|T\|$, T_1+T_2, T_1T_2 as usual, and have the relations $\|Tf\| \leqq \|T\| \cdot \|f\|$, $T_1(T_2+T_3) = T_1T_2+T_1T_3$, $(T_1+T_2)T_3 = T_1T_3+T_2T_3$. Suppose now we have a subset $\mathfrak{T}^*$ of $\mathfrak{T}$, which contains along with any two elements their sum and product. Suppose each element $T \epsilon \mathfrak{T}^*$ has an inverse $T^{-1} \epsilon \mathfrak{T}$; thus each element of $\mathfrak{T}^*$ is a one-to-one transformation of $\mathfrak{B}$ into itself. The transformation 0, which carries any element into the zero element, may however be included in $\mathfrak{T}^*$ and is an exception to the last statement. Suppose further that there exists a constant W, depending only on $\mathfrak{T}^*$, such that if for any elements $T_1, \cdots, T_n$ of $\mathfrak{T}^*$ with $T_1 + \cdots + T_n \neq 0$, we set $T_i' = T_i(T_i + \cdots + T_n)^{-1}$ so that $T_1' + \cdots + T_n' = I$, the relation

$$\|\Sigma\, T_i' f_i\| \leqq W \sup \|f_i\|.$$

holds.

For example, let $\mathfrak{B}$ be the space of matrices

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

with norm defined by $(\Sigma a^2_{rs})^{1/2}$. Let $\mathfrak{T}^*$ be the set of all transformations sefined by

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

as a multiplier on the left, where α and β are real and non-negative. It is easily verified that for operations T of this class, $\|T\|$ is the greater of α, β. Then if $\Sigma_1^n T_i = I$, we have

$$\begin{aligned} \|\Sigma T_i f_i\| &\leqq \Sigma \|T_i f_i\| \leqq \Sigma \|T_i\| \cdot \|f_i\| \\ &\leqq \sup \|f_i\| \cdot \Sigma \|T_i\| \leqq 2 \sup \|f_i\|. \end{aligned}$$

Thus we see that $W \leqq 2$.

Suppose now a transformation $T(A)f$ is assigned to each set A of $\mathfrak{A}$ in such a way that $T(A)$ has the properties of a completely additive measure. We shall assume that if any set has measure zero, the same holds for any of its subsets. If we define as in section 2,

$$S(\mathcal{P}) = \Sigma_i T(p_i) f(x_i),$$

$$D(\mathcal{P}) = \Sigma_i T(p_i) [f(x_i') - f(x_i'')],$$

$$\omega(\mathcal{P}) = \sup \| D(\mathcal{P}) \|,$$

the whole of the work of sections 2, 3, 4 can be carried through with only the obvious changes of replacing $m(p)$ by $T(p)$, $1/m(p)$ by $[T(p)]^{-1}$ throughout, and introducing the constant W where necessary. In the lemma, for example, we have $\|D(Q)\| \leqq W\omega(\mathcal{P})$, and in Theorem 3(g) we have $\|K(f,P)\| \leqq MW\|T(P)\|$. In section 3, statements such as $m(P-B) < \delta$ are replaced by $\|T(P-B)\| < \delta$. The details involved in carrying out this program are left to the reader.

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A Contour Integral and First Order Expansion Problem

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The purpose of this paper is to extend the scope of a certain contour integral used in the convergence proofs of various expansion problems and to illustrate its use in one simple case involving the solutions of a particular first order differential equation.

I. The contour integral

$$(1)\qquad \lim_{m=\infty} \frac{1}{2\pi i}\int_{\Gamma_m} \frac{e^{-pcz}dz}{(1-e^{-cz})z^k}$$

$$= \left\{\begin{array}{ll} 1/2, & (p=0), \\ 0, & (0<p<1, \text{ and real}), \\ -1/2, & (p=1), \end{array}\right\} (k=1),$$

has been used many times. In this paper it is proposed to extend the range of p to any value, real or complex, and to allow k to be any positive integer. In this case there is a $(k+1)$-pole at the origin, with other poles symmetrically spaced along the imaginary axis at the points $z=2\pi mi/c$, $(m=\pm 1, \pm 2, \cdots)$. The contour Γ_m may be any curve which encloses the imaginary axis but passes midway between two successive poles and recedes indefinitely with m. For example, circles with centers at the origin and radii equal to $2\pi(m+1/2)/c$ may be used.

If the integrand is expanded in the power series

$$(2)\qquad \frac{e^{-pcz}}{(1-e^{-cz})z^k} = \frac{h}{z^{k+1}} + \cdots + \frac{b}{z^2} + \frac{a}{z} + s + tz + \cdots,$$

"a" will be the residue for the $(k+1)$-pole at the origin. To determine its value, multiply by z^{k+1}, expand the left side in the known power series in z and equate the coefficients of z^k. Whence

$$(3)\qquad a=(-1)^k\frac{c^{k-1}}{k!}\left\{ B_k(p) - \frac{(-1)^{k/2}}{2}[1+(-1)^k]B_{k/2}\right\}, \quad (k\geqq 2),$$

where $B_k(p)$ and B_k are Bernoulli polynomials and numbers [1]. When $k=1$, $a=-B_1(p)+1/2$.

For the other poles the residues will be

$$(4)\qquad \lim_{z=2\pi mi/c} \frac{(z-2\pi mi/c)e^{-pcz}}{(1-e^{-cz})z^k}$$

$$= \frac{e^{-2pm\pi i}}{c(2\pi mi/c)^k}, \ (p \text{ real}, \ m=\pm 1, \ \pm 2, \cdots).$$

Summing up these residues by pairs for poles which are symmetrically placed with respect to the origin, we obtain [2]

$$(5)\qquad \left.\begin{array}{l} (-1)^{k/2}(c/2)^{k-1}\displaystyle\sum_{m=1}^{\infty} \frac{\cos 2\pi mp}{(\pi m)^k} \ (k \text{ even}) \\ (-1)^{(k+1)/2}(c/2)^{k-1}\displaystyle\sum_{m=1}^{\infty} \frac{\sin 2\pi mp}{(\pi m)^k} \ (k \text{ odd}) \end{array}\right\}$$

$$= (-1)^{k+1}\frac{c^{k-1}}{k!}\left\{ B_k(p) - \frac{(-1)^{k/2}}{2}[1+(-1)^k]B_{k/2} \right\}, \ (k \geqq 2, \ 0<p<1),$$

and
$$-\sum_{m=1}^{\infty} \frac{\sin 2\pi mp}{\pi m} = B_1(p) - 1/2, \quad (k=1).$$

The total residue and the value of the integral is thus zero on adding the residues for $(0<p<1)$ for all positive values of k, which agrees with the original integral. Nörlund's formulas [2] are changed to agree with the definitions of $B_k(p)$ given by Bromwich [1].

But it is known [3] that

$$(6)\qquad 1/\pi\sum_{m=1}^{\infty} \frac{\sin 2\pi mp}{m} = \begin{cases} 0, & (p \text{ integral}, \\ -B_1(p)+1/2+[p], & (p \text{ not integral}), \end{cases}$$

where $[p]$ designates the largest integer algebraically less than p. By successive integration it is found [3] that

$$(7)\qquad \left.\begin{array}{l} (-1)^{(k+1)/2}(c/2)^{k-1}\displaystyle\sum_{m=1}^{\infty} \frac{\sin 2\pi mp}{(\pi m)^k} \ (k \text{ odd}) \\ (-1)^{k/2}(c/2)^{k-1}\displaystyle\sum_{m=1}^{\infty} \frac{\cos 2\pi mp}{(\pi m)^k} \ (k \text{ even}) \end{array}\right\}$$

$$= (-1)^k c^{k-1}\left\{ \frac{-B_k(p) + \dfrac{(-1)^{k/2}}{2}[1+(-1)^k]B_{k/2}}{k!} \right.$$

$$+\sum_{n=0}^{k-1} \frac{(p-[p])^n B_{k-n}([p])}{n!\,(k-n)!} \Bigg\}, \quad (k \geqq 2).$$

Finally, adding these residues to that obtained for the pole at the origin, the value of the integral is found to be

$$\text{(8)} \qquad \operatorname*{Lim}_{m=\infty} \frac{1}{2\pi i} \int_{\Gamma_m} \frac{e^{-pcz}dz}{(1-e^{-cz})z^k}$$

$$= (-1)^k c^{k-1} \sum_{n=0}^{k-1} \frac{(p-[p])^n B_{k-n}([p]).}{n!(k-n)!}\,.$$

which holds for $k \geqq 2$ with p any real value. If $k=1$,

$$\text{(9)} \quad \operatorname*{Lim}_{m=\infty} \frac{1}{2\pi i} \int_{\Gamma_m} \frac{e^{-pcz}dz}{(1-e^{-cz})z} = \begin{cases} -B_1(p)+1/2, & (p \text{ integral}), \\ -B_1([p]), & (p \text{ not integral}). \end{cases}$$

When p is complex of the form $a+bi$, $(b \neq 0)$, the residues

$$\frac{e^{-2\pi pmi}}{2(\pi mi)^k} = \frac{e^{-2\pi mia}e^{2\pi mb}}{2(\pi mi)^k}\,.$$

The absolute value of the latter form has an unbounded limit as m becomes infinite, and thus the series of residues does not converge, with the result that the integral itself does not have a limit.

II. The above integral with $k=1$ but with p extended beyond the region from zero to one may be applied to the convergence of an expansion of an arbitrary function in terms of the solutions of a certain simple differential equation of the first order.

Consider the equation

$$\text{(10)} \qquad dy/dx + \lambda x^h y = 0, \quad (h>0),$$

with the boundary conditions $y(-a)=y(b)$. Introducing the new parameter $\nu=\lambda A$, where A is the average value of x^h integrated from $-a$ to b, the principal parameter values of ν are found to be

$$\text{(11)} \qquad \nu^* = 2\pi mi/(a+b), \quad (m=0, \pm 1, \pm 2, \cdots),$$

which are symetrically placed on the imaginary axis in the ν-plane.

Assuming the expansion

$$f(x) = \sum C_m y, \quad (m=0, \pm 1, \pm 2, \cdots),$$

where $f(x)$ is suitably restricted, C_m is determined by using the adjoint equation and the orthogonality condition and the expansion becomes [4]

$$(12) \qquad f(x) = \sum \int_a^b f(u) u^h e^{\lambda \int_x^u x^h dx} du/(a+b)A,$$

provided $A \neq 0$.

Green's function in this case is found to be

$$(13) \qquad G(x,\lambda) = \begin{cases} e^{\lambda \int_x^u x^h dx}/A[1-e^{-(a+b)\nu}], & (u<x), \\ e^{\lambda \int_x^u x^h dx}/A[e^{(a+b)\nu}-1], & (u>x), \end{cases}$$

the residue of which in the ν-plane is

$$e^{\lambda \int_x^u x^h dx}/(a+b)A.$$

The residue of $$\int_{-a}^b f(u)G(x,\lambda)u^h du$$

is then $$\int_{-a}^b f(u)u^h e^{\lambda \int_x^u x^h dx} du/(a+b)A,$$

which is a term of the expansion. The convergence of the series may be demonstrated by the use of Birkhoff's Contour Integral Method [4], [5], in the application of which it is necessary to evaluate the contour integral

$$(14) \quad \lim_{m=\infty} \frac{1}{2\pi i} \int_{\Gamma_m} \int_{-a}^b f(u)u^h G(x,\lambda) du d\nu$$

$$= \lim_{m=\infty} \frac{1}{2\pi i} \int_{\Gamma_m} \left\{ \int_{-a}^x \frac{f(u)u^h e^{\lambda \int_x^u x^h dx} du}{A(1-e^{-(a+b)\nu})} + \int_x^b \frac{f(u)u^h e^{\lambda \int_x^u x^h dx} du}{A(e^{(a+b)\nu}-1)} \right\} d\nu,$$

where Γ_m is a circle in the ν-plane with its center at the origin and with its radius equal to $2\pi(m+\frac{1}{2})/(a+b)$, increasing indefinitely with m.

Integrating by parts, the following expression is obtained,

$$(15) \quad \lim_{m=\infty} \frac{1}{2\pi i} \int_{\Gamma_m} \left\{ \frac{f(x^-) - f(-a)e^{\lambda \int_x^{-a} x^h dx} - \int_{-a}^x f'(u) e^{\lambda \int_x^u x^h dx} du}{(1-e^{-(a+b)\nu})\nu} \right.$$

$$+\frac{f(b)e^{\lambda\int_x^b x^h dx}-f(x^+)-\int_x^b f'(u)e^{\lambda\int_x^u x^h dx}du}{(e^{(a+b)\nu}-1)\nu}\Bigg\}d\nu,$$

where x^- and x^+ indicate that x is approached from the left or right respectively.

The contour integration of the terms involving $f(x)$ produces in the limit the expression $\frac{1}{2}[f(x^-)+f(x^+)]$, which is the mean value of $f(x)$ at x. In evaluating the remaining integrals different cases depending on the value of h will have to be considered.

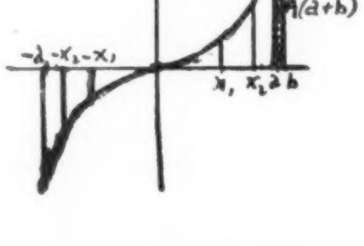
(a) When h is even or of such fractional value that x^h is positive throughout the interval, the exponential factors of $f(-a)$ and $f(b)$ will have the form $e^{\pm p(a+b)\nu}$, $(0<p<1$, and the contour integrals approach zero in the limit (1). In the two remaining integrals take absolute values, remove the exponential by the first mean value theorem and integrate with respect to u directly. The same type of exponentials will be present and these terms will also contribute zero in the limit. Therefore, in this case the expansion converges.

(b) When h is of such fractional value that xj is complex, the coefficients of ν in the exponentials will be complex also, and the integrals considered become infinite and the series diverges as it was indicated in the last paragraph of (I).

(c) When h is odd or of such value that x^h changes sign in passing through the origin, the subcases where $a \lesseqgtr b$ must be considered.

(c$_1$) When $a<b$, $\displaystyle\int_{-a}^{b} x^h dx = A(a+b) > 0$. (Shaded area).

When $x>a$, both $\displaystyle\int_{-a}^{x} x^h dx$ and $\displaystyle\int_{x}^{b} x^h dx$



will be positive and less than $A(a+b)$ and the contour integrals involving them will approach zero in the limit as usual. But when $-a<x<a$,

$$\int_{-a}^{x} x^h dx$$

may, for a finite number of values of x, be a negative integral multiple p $(0<p<P$, where P is finite) of $A(a+b)$, while at all other points of the interval it is negative but not an integral multiple of $A(a+b)$. Also for the same finite set of points,

$$\int_{x}^{b} x^h dx$$

will be a positive multiple $(p+1)$ of $A(a+b)$, and a fractional multiple though larger by unity at all other points, as before. The lower limit terms therefore have the following contour integrals.

$$\text{(16)}\quad \operatorname*{Lim}_{m=\infty} \frac{f(b)}{2\pi i} \int_{\Gamma_m} \frac{e^{\nu(a+b)(p+1)} d\nu}{(e^{(a+b)\nu}-1)\nu}$$

$$- \operatorname*{Lim}_{m=\infty} \frac{f(-a)}{2\pi i} \int_{\Gamma m} \frac{e^{\nu(a+b)p} d\nu}{(1-e^{-(a+b)\nu})\nu}$$

$$= \operatorname*{Lim}_{m=\infty} \frac{f(b)-f(-a)}{2\pi i} \int_{\Gamma m} \frac{e^{-\nu(a+b)(-p)} d\nu}{(1-e^{-(a+b)\nu})\nu}$$

$$= [f(b)-f(-a)] \cdot \begin{cases} \frac{1}{2}+p, & (p \text{ integral}), \\ -[-p], & (p \text{ not integral}), \end{cases}$$

by equation (9). If $f'(u)$ is such a function that the remaining terms cannot be integrated directly with respect to u, it may be replaced by an upper bound. The absolute values of the "u" integrals taken between the given finite limits have an upper bound, and thus the contour integrals involving them are bounded. When $f'(u)$ is such a function that these "u" integrals may actually be integrated directly with respect to u, it is found that the contour integrals either approach zero or some bounded function of $f(x)$, i. e., of $f(b)$ and $f(-a)$, at the upper and lower boundaries.

(c$_2$) When $a>b$, $$\int_{-a}^{b} x^h dx = A(a+b) < 0.$$

When $x < -b$, the contour integrals will approach zero as above. When $-b < x < b$, the lower limit contour integrals become

$$\text{(17)} \quad \lim_{m=\infty} \frac{f(b)}{2\pi i} \int_{\Gamma_m} \frac{e^{-\nu(a+b)p} d\nu}{(e^{(a+b)\nu}-1)\nu}$$

$$- \lim_{m=\infty} \frac{f(-a)}{2\pi i} \int_{\Gamma m} \frac{e^{-\nu(a+b)(p+1)} d\nu}{(1-e^{-(a+b)\nu})\nu}$$

$$= \lim_{m=\infty} \frac{f(b)-f(-a)}{2\pi i} \int_{\Gamma m} \frac{e^{-\nu(a+b)(p+1)} d\nu}{(1-e^{-(a+b)\nu})\nu}$$

$$= [f(b)-f(-a)] \cdot \begin{cases} -(p+\frac{1}{2}), & (p \text{ integral}), \\ -[p+1], & (p \text{ not integral}). \end{cases}$$

$$(c_3) \quad \text{When } |a-b| \to 0, \quad \int_{-a}^{b} x^h dx = A(a+b) \to 0,$$

and the problem may be handled as a limiting case of (c_1) or (c_2). As $a \to b$, the values of x where p is an integer will become closer and closer together and $p \to \infty$. In the limit, $A=0$, and $p=\infty$ everywhere, and the term added to $f(x)$ becomes infinite.

It may thus be concluded that when x^h is positive the expansion converges uniformly to $f(x)$ throughout the interval $-a < x < b$, when x^h is complex the expansion diverges, and when x^h changes sign and $a \neq b$ the expansion converges to $f(x)$ plus additional terms involving the end points of the interval. When $a=b$ the expansion also diverges.

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The Teachers' Department

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Coordination of Mathematics and Science Through Student Activities

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The writing of this article was prompted by the enthusiastic reception of the ideas outlined herein, not only by the students, but also by several members of the faculty of Brooklyn College.

In centuries gone by the teacher, scientist, and philosopher were usually one and the same person and the student got the benefit of not only a drill in a specific doctrine or branch of science but also an insight into the interrelation of things, a general outlook or "Weltanschauung" as the Germans aptly put it. The tremendous development of the various fields of scientific endeavor in modern times has brought about an overspecialization and an almost complete separation of the teaching of each branch of natural and abstract science.

The important role of mathematics in the great strides of development of the scientific thought is well known and recognized, as evidenced by the following quotations of eminent scientists. Sir William Bayliss* states: "... The ultimate aim of all science is to express in a mathematical form the discoveries that have been made...". Bell† writes: "... Others with a keener scientific insight have actually predicted, from their sheer mathematics, physical phenomena which until thus predicted had not even been suspected, much less observed. Prediction is the most characteristic service which mathematics renders the sciences." Salpeter‡ remarks: The naturalist gains from the study of mathematics a definite mental training, a

*Feldman, W. M., *Biomathematics, Introduction*, p. 10. London, Charles Griffin and Company, Ltd., 1923.

†Bell, E. T., *The Handmaiden of the Sciences*, Baltimore, Williams & Wilkins, 1937, p. 2.

‡Salpeter, J., *Einfuehrung in die Hoehere Mathematik fuer Naturforscher und Aerzte*, p. IV, Jena, Gustav Fisher, 1926.

precision of the methods of thinking." Courant* states: "The isolation of the essential mathematical features from the physical features of a given problem often exhibits the core of the problem and shows apparently different and independent phenomena may have identical underlying structures...".

Nevertheless the present textbooks and methods of teaching mathematics in high schools and colleges evoke a great deal of complaints on the part of prominent scientists in the other fields. Thus Feldman† quoting a foremost British physiologist who said: "So much of what is evidently important is veiled in mathematical language'—concludes that "...the time is, therefore, ripe for a book which should explain to the biological student those portions of the so-called higher mathematics which are now being utilized in the study and investigations of biological problems." Mellor‡ states: "It is almost impossible to follow the later developments of physical or general chemistry without a working knowledge of higher mathematics. I have found that the regular textbooks of mathematics rather perplex than assist the chemical student." Von Kármán and Biot§ remark: "... the amount of mathematics included in the curriculum is quite adequate, but the ability to find the proper mathematical setup for given physical or engineering problems is not developed in the student to a sufficient degree... the need is not so much for 'more mathematics' as for a better understanding of the potentialities of its application."

In an effort to correct these conditions a large number of textbooks were published and, in some institutions, special courses were introduced, such as "Mathematics in Biology," "Mathematics in Chemistry," etc. Excellent as these texts⁑ may be, from the point of view of contents or arrangement, they still do not fulfill the need of the student or the teacher for several reasons. In the first place, mathematics is treated as if it were a tool, and too much mathematics is condensed in a comparatively few pages. There is too much emphasis on the solution of numerous problems in the particular field of applications, but not enough theory to enable the student to grasp the fundamental structure of the problem. In the second place, the process of bridging or coordination is started on too high a level so

*Courant, R., *Advanced Methods in Applied Mathematics*, p. 1, New York University Lectures, 1941.

†*loc. cit.*, Preface.

‡Mellor, J. W., *Higher Mathematics for Students of Chemistry and Physics*, p. IX, Longmans, Green and Company, 1905.

§v. Kármán and Biot *Mathematical Methods in Engineering*, p. V. New York and London, McGraw Hill Book Co., Inc., 1940.

⁑Several such textbooks are listed at the end of the article

that even a student familiar with both mathematics and the particular field of science finds it difficult to use the text. Some of the authors of such texts realize the inadequacy of the mathematical material in their books. For instance Daniels* says: "If the student is satisfied with this book, the author will be disappointed, for it is his hope that many who read these pages will resolve to go further into the study of mathematics."

As a result of many years of experience and experimentation with courses based on such textbooks, as well as a continued and active part in student activities, I have come to the conclusion that an entirely different approach is necessary if we want to achieve a real and organic coordination of the various scientific doctrines in our system of education. It is the purpose of this article to outline the principles of this approach and the reasons which make this method superior to the introduction of special courses.

The suggested approach consists in the utilization of students' extra-curricular activities, on a voluntary basis, by helping them to form clubs or societies dedicated to this type of work. It can and should be started as early as possible, preferably in the high school. The club or society should be under the general supervision of a member of the teaching staff of the Department of Mathematics with co-advisory assistance of the faculties of other departments, such as biology, chemistry, and physics. This advisory body prepares a list of topics from which each student selects a subject in which he is interested and upon which he desires to work in preparation of a paper to be presented to the club members. One topic may be selected by two or three students who are then encouraged to work together. In the course of their work they get the benefit of the advice of both the faculty adviser and the particular co-adviser in whose field the topic belongs. Before presentation the report is reviewed by the advisers in order to insure the correctness, completeness, and clarity of presentation. Whenever more than one student works on a topic the presentation may be made by one of them, or it may be divided into parts so that each collaborator presents a separate phase of the subject.

It is, of course, the responsibility of the faculty adviser to arrange the topics so that their order forms a logical and organic sequence. The faculty adviser must be present at the meetings and be able to answer questions that may come both from the students who present the topic and from those listening. Above all, he or she must make sure that the form and manner of presentation is such that it will create and sustain the interest of all who attend the meeting.

*Daniels, F., *Mathematical Preparation for Physical Chemistry*, New York, McGraw-Hill, 1928. Introduction.

Let us now examine the advantages of this program and the reasons why it will achieve the desired objectives. The most important element of the program is the fact that the presentation of the topics and the underlying research is done by the students themselves. In this manner they learn to be self-reliant, and in the effort to find out things for themselves, discipline their own minds and develop independence of thought. Whenever two or more students work together they develop a spirit of cooperation. If one of them happens to be a mathematics major he derives the benefit of broadening his horizon by the study of applications in other fields, while the others have an opportunity to fill the gaps in their mathematical training. The presentation of the same mathematical principle applied to a number of diverse problems in several scientific fields brings out not only the significance of pure theory but also a better understanding of the particular problem and a facility of analyzing the nature of the problem and recognizing the essential elements thereof.

Since the program is started at an early stage, it permits many students to become aware of what the various fields of science really are like and thus aids them in their self-orientation and selection of the field in which they want to major. The more intimate contacts between students and teachers permit a better evaluation of the students' latent abilities and thus aid in their development by proper encouragement and guidance. They stimulate both student and teacher and create a bond between them which can rarely be achieved in the classroom.

Finally, a great advantage lies in the cooperation between the faculties of the several departments involved. This type of cooperations permits the closer coordination of the teaching of the courses given in the various departments and the elimination of needless duplication. It leads to a better understanding of the requirements of such departments and therefore to basic reforms in the curriculum of studies, the arrangement of the material in the individual courses, and even in methods of presentation. Lest I am misunderstood, I wish to emphasize that I do not advocate any diminution of the teaching of pure theory in favor of extended applications. On the contrary, the need is to have each department present its doctrines as fully and independently as possible. Mathematics, if it is to continue to play its significant part in liberal education—that is, in disciplining the mind, in training the student to think independently and to realize the value of postulational thinking—should never be presented merely as a tool to solve a particular problem.

For many years I have kept in touch with the students even after they have left college and entered various fields of endeavor and from

these contacts I have become convinced that it is not facts or knowledge that the students lack. What they do not possess is the facility to recognize where and when they should use the particular knowledge which they have. It is therefore coordination which they require. The interdepartmental program outlined above provides, in my opinion, the best means of integrating the various phases of education and thus gives the students a better preparation for their future tasks or professions.

While the central subject in the preceding discussion was the coordination of mathematics and science, the program is equally well suitable to other fields. For example, one can bring together history, literature, social science and geography; or art, drama and music. The key to the success of the program is the cooperation among the faculty members of the various departments involved and their ability to inspire students with a desire to continue to work under their guidance. It is my intention to prepare one or more additional articles dealing with the details of this program and various problems which arise in this connection. I would, therefore, welcome any criticisms, suggestions, or questions, for they will help in the perfection of the various phases of the program.

LIST OF SPECIAL TEXTBOOKS

Allen, R. G. D., *Mathematical Analysis for Economists*, London, Macmillan and Company, 1938.

Feldman, W. M., *Biomathematics*, London, Charles Griffin and Company, 1923.

Mellor, J. W., *Higher Mathematics for Students of Chemistry and Physics*. Longmans, Green and Company, 1905.

Partington, J. R., *Higher Mathematics for Chemical Students*. New York, D. Van Nostrand Co., 1912.

Salpeter, J., *Einfuehrung in die Hoehere Mathematik fuer Naturforscher und Aerzte*, Jena, G. Fisher, 1926.

Sherwood and Reed, *Applied Mathematics in Chemical Engineering*, New York, McGraw-Hill Book Co., Inc., 1939.

Bill of Rights of Teachers of Secondary Mathematics

EDITORIAL NOTE. The following communication has been received from the Chairman of the MATHEMATICS INSTITUTE of Duke University. We believe that our readers will find it a cogent, worthwhile statement.

A Bill of Rights for the teacher of mathematics has a two-fold purpose. First it should declare his right to an *opportunity* for adequate preparation for the tasks which lie ahead. Second it should set forth his right to share fully in the *responsibility* associated with being a teacher of mathematics.

The nature of mathematics and its uses in the work-a-day world can be made of primary significance. This can be done by facing the facts with intelligence, courage, and patience. There is a desperate need for a meaningful understanding of relationships (both arithmetical and functional) as well as of the techniques of mathematical manipulation. The quantitative aspect of our current living offers both the opportunity and the responsibility for a fuller understanding through the mode of analysis commonly called mathematics.

Accordingly we believe a teacher of secondary mathematics has two sets of rights:

A. *Those relating to opportunity:*

1. To expect colleges and universities to offer mathematics courses of more functional value than is being done.
2. To study at first-hand the applications of mathematics to science, engineering, social science, and government under competent instructors.
3. To have experience in business, in industry, and in government in order to become familiar with current practices in applications of mathematics.
4. To expect that school boards will provide financial assistance for teachers to visit other schools and attend conferences and institutes.
5. To expect encouragement from school administrators for needful experimentation with recent developments in content materials and methods of instruction.
6. To expect a salary commensurate with his training and his responsibilities.
7. To have access to a mathematics laboratory with its library, illustrative devices, mathematical instruments, and teaching aids for classroom use.
8. To participate in curriculum building and adaptation of the curriculum to his students in mathematics and in the selection of textbooks.
9. To have satisfactory tenure provisions, and adequate certification standards.

B. *Those relating to responsibility:*

1. To acquire the knowledge and the skills needed in assisting students to understand and appreciate mathematics.
2. To become familiar with the vocational opportunities in his field and in related fields in order to guide students intelligently.
3. To see that students realize the broad objectives essential to good citizenship and satisfactory vocational performances.
4. To help establish and maintain high standards of excellence in teaching.
5. To encourage students to broaden their horizons by investigating quantitative relationships wherever they be found.
6. To participate cooperatively in the best available in-service training.
7. To be familiar with the historical development of mathematics and its uses through the ages. (Such knowledge has both cultural and utilitarian values).
8. To affiliate with such organizations as promote the study of mathematics on the secondary level and stimulate his professional growth.

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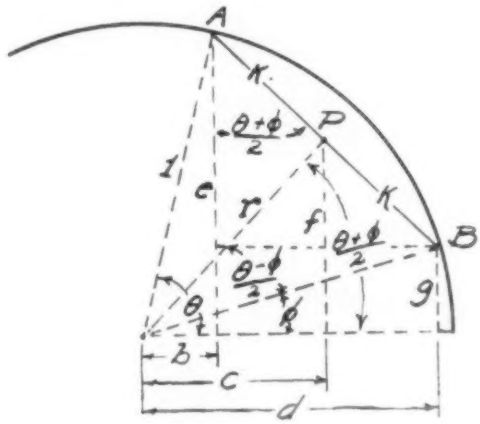
Mathematics Institute,
Duke University,
July 3-12, 1945.

Brief Notes and Comments

Edited by
MARION E. STARK

20. *Derivation of the product formulas of trigonometry from a figure.* Consider a chord of the unit circle as shown in the figure. Then



$$e=\sin\theta \qquad g=\sin\phi \qquad f=r\sin\frac{\theta+\phi}{2} \qquad k=\sin\frac{\theta-\phi}{2}$$

$$b=\cos\theta \qquad d=\cos\phi \qquad c=r\cos\frac{\theta+\phi}{2} \qquad r=\cos\frac{\theta-\phi}{2}$$

Since P is the midpoint of AB,

$$e+g=2f=2r\sin\frac{\theta+\phi}{2}$$

$$\sin\theta+\sin\phi=2\cos\frac{\theta-\phi}{2}\sin\frac{\theta+\phi}{2} \tag{1}$$

$$e-g=2k\cos\frac{\theta+\phi}{2}$$

$$\sin\theta-\sin\phi=2\sin\frac{\theta-\phi}{2}\cos\frac{\theta+\phi}{2} \tag{2}$$

$$d+b=2c=2r\cos\frac{\theta+\phi}{2}$$

$$\cos\phi+\cos\theta=2\cos\frac{\theta-\phi}{2}\cos\frac{\theta+\phi}{2}.$$

$$d-b=2k\sin\frac{\theta+\phi}{2},$$

$$\cos\phi-\cos\theta=2\sin\frac{\theta-\phi}{2}\sin\frac{\theta+\phi}{2}. \tag{4}$$

Having determined these geometrically we have only to let $\phi=0$ in identities (1), (3) and (4) to derive the following standard formuas:

$$\sin\theta=2\cos\frac{\theta}{2}\sin\frac{\theta}{2},$$

$$\cos\theta+1=2\cos^2\frac{\theta}{2} \quad \text{or} \quad \cos\frac{\theta}{2}=\sqrt{\frac{1+\cos\theta}{2}}$$

$$1-\cos\theta=2\sin^2\frac{\theta}{2}, \text{ or } \sin\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{2}}.$$

U. S. Navy. LT. (j.g.) P. D. THOMAS.

Problem Department

Edited by
E. P. STARKE and N. A. COURT

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscripts be typewritten with double spacing. Send all communications to E. P. STARKE, Rutgers University, New Brunswick, N. J.

SOLUTIONS

No. 580.* Proposed by *Howard Eves*, College of Puget Sound.

The trilinear polars of the isotomic conjugates of any two points collinear with the centroid of the triangle are parallel.

I. Solution by *Joseph S. Guérin*, Catholic University of America.

Given a triangle $(T)=ABC$, let E, F be two points collinear with the centroid G of (T) and let E', F' be the isotomic conjugates of E, F for (T).

Let P, Q; R, S; M, N; P', Q'; R', S' be the feet of the cevians BEP, CEQ; BFR, CFS; BGM, CGN; $BE'P'$, $CE'Q'$; $BF'R'$, $CF'S'$.

If D is the trace of the line EFG on BC, we have:

$$(P'R'MA)\overline{\wedge}(PRMC)\overline{\wedge}B(PRMC)\overline{\wedge}(EFGD)\overline{\wedge}C(EFGD)$$

$$\overline{\wedge}(QSNB)\overline{\wedge}(Q'S'NA).$$

Hence, considering the biratios (i. e., the anharmonic ratios) of the extreme terms of this projectivity, we have

$$(1) \qquad P'M\cdot AR'/P'A\cdot MR'=Q'N\cdot AS'/Q'A\cdot NS'.$$

Let the trilinear polars $P''Q''$, $R''S''$ of the points E', F' meet the sides AC, AB in the pairs of points P'', R''; Q'', S''. We have (see Court's *College Geometry*, p. 137, Art. 257):

$$P'A\cdot P'C=P'P''\cdot P'M,$$

or, putting $P'C=P'M+MC$, $P'P''=P'A+AP''$,

$$P'A\cdot MC=AP''\cdot P'M.$$

*October, 1944.

Similarly we have:

$$R'A \cdot MC = AR'' \cdot R'M$$

The result obtained from cross-multiplying the last two equalities may be put in the form

$$P'M \cdot AR'/P'A \cdot MR' = AR''/AP''. \tag{2}$$

Considering the points on the side AB we obtain, in a like manner,

$$Q'N \cdot AS'/Q'A \cdot NS' = AS''/AQ'' \tag{3}$$

From (1), (2), and (3) we have

$$AR'' : AP'' = AS'' : AQ'',$$

hence the two lines $P''Q''$, $R''S''$ are parallel.

II. Solution by *L. M. Kelly*, U. S. Coast Guard Academy.

We make use of barycentric coordinates and employ the following standard results,

a) The trilinear polar of the point $(m_1m_2m_3)$ is

$$\frac{x_1}{m_1}+\frac{x_2}{m_2}+\frac{x_3}{m_3}=0.$$

b) The isotomic conjugate of $(m_1m_2m_3)$ is

$$\left(\frac{1}{m_1}, \frac{1}{m_2}, \frac{1}{m_3}\right)$$

c) If two lines $\begin{cases} a_1x_1+a_2x_2+a_3x_3=0 \\ b_1x_1+b_2x_2+b_3x_3=0 \end{cases}$

are parallel, then $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 1 & 1 & 1 \end{vmatrix}=0$

and conversely.

d) if points $(a_1a_2a_3)$, $(b_1b_2b_3)$, $(c_1c_2c_3)$ are collinear then

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=0$$

and conversely.

Now let $P(p_1p_2p_3)$, and $Q(q_1q_2q_3)$ be the points collinear with the centroid $M(1,1,1)$. Their isotomic conjugates will be

$$\left(\frac{1}{p_1} \quad \frac{1}{p_2} \quad \frac{1}{p_3}\right) \text{ and } \left(\frac{1}{q_1} \quad \frac{1}{q_2} \quad \frac{1}{q_3}\right)$$

respectively. The trilinear polars of these two points will, by result a) above, be

$$p_1x_1+p_2x_2+p_3x_3=0 \quad \text{and} \quad q_1x_1+q_2x_2+q_3x_3=0.$$

These will be parallel if $\begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ 1 & 1 & 1 \end{vmatrix}=0,$

which condition is satisfied, since P,Q,M are collinear.

Also solved, in a similar way, by the *Proposer* who supplemented his solution by the following remarks:

Let (p_1,p_2,p_3) be the areal coordinates of any point P. Let P' be the point $(k_1/p_1,k_2/p_2,k_3/p_3)$, where k_1, k_2, k_3 are fixed constants. Then we may call P' the *associated conjugate* of P for the constants k_1, k_2, k_3. The trilinear polar, p', of P' may be called the *associated polar* of P, and P the *associated pole* of p'. If P and P' coincide, P will be called a *self-conjugate* point. Now, along the lines indicated above, it is easy to prove that

1. The associated polars of a set of collinear points are concurrent, and dually, the associated poles of a set of concurrent lines are collinear.

2. The associated polars of two points collinear with a self-conjugate point meet on the trilinear polar of that self-conjugate point.

3. The common point of a set of concurrent associated polars is the associated pole of the line of collinearity of their associated poles, and dually.

4. Cross-ratio is invariant under an *associate correlation.*

5. The triangle determined by three points P, Q, R is perspective with the triangle determined by their three associated polars p', q', r'.

6. The locus of points lying on their own associated polars is the conic S: $x_1^2/k_1+x_2^2/k_2+x_3^2/k_3=0$.

7. A point and its associated polar are pole and polar with respect to the conic S.

8. A point and its isotomic conjugate are associated conjugates for the constants 1, 1, 1. A point and its isogonal conjugate are associated conjugates for the constants a_1^2, a_2^2, a_3^2, where a_1, a_2, a_3 are the sides of the triangle of reference.

No. 580.* Proposed by *Howard Grossman*, New York City.

$$\text{Find} \quad \lim_{n\to\infty} \cos\frac{\pi}{3}\cos\frac{\pi}{4}\cos\frac{\pi}{5}\cdots\cos\frac{\pi}{n}.$$

(See Kasner's *Mathematics and the Imagination*, p. 311, for an interesting geometric interpretation.)

II. Solution by *H. E. Fettis*, Dayton, Ohio.

An expression in terms of known constants which converges much more rapidly than the method of the Note in this MAGAZINE, May, 1945, p. 424, may be obtained in the following way. Let

$$P=\lim_{n\to\infty} \cos\frac{\pi}{3}\cos\frac{\pi}{4}\cos\frac{\pi}{5}\cdots\cos\frac{\pi}{n},$$

whence we have
$$\log P=\sum_{n=3}^{\infty}\log\cos\frac{\pi}{n}.$$

From the known series

$$-\log\cos x=\sum_{k=1}^{\infty}\frac{2^{2k-1}(2^{2k}-1)\,B_{2k-1}x^{2k}}{k(2k)!},$$

where B_{2k-1} are the Bernoulli numbers, we may write

$$-\log\cos\frac{\pi}{n}=\sum_{k=1}^{\infty}\frac{2^{2k-1}(2^{2k}-1)B_{2k-1}\pi^{2k}}{k(2k)!\,n^{2k}}$$

Further, if $\zeta(x)$ is the Zeta function of Riemann, we have

$$2^{2k-1}B_{2k-1}\pi^{2k}=\zeta(2k)(2k)!$$

and
$$\zeta(2k)=\sum_{n=1}^{\infty}1/n^{2k}, \quad\text{or}\quad \sum_{n=3}^{\infty}1/n^{2k}=\zeta(2k)-1-1/2^{2k}.$$

Combining the above chain of results we have

$$-\log P=\sum_{k=1}^{\infty}\frac{(2^{2k}-1)\zeta(2k)}{k}\left[\zeta(2k)-1-1/2^{2k}\right].$$

The values of $\zeta(x)$ may be found from tables, or may be computed directly: $\zeta(2)=\pi^2/6$, $\zeta(4)=\pi^4/90$, $\zeta(6)=\pi^6/945$, $\zeta(8)=\pi^8/9450$, $\zeta(10)=\pi^{10}/93555$; and for $k>5$, the series for $\zeta(x)$ converges rapidly. From ten terms of the series for $-\log P$ we obtain without difficulty.

$$\log P=-2.16333, \quad\text{whence}\quad P=.11495.$$

*November, 1944.

No. 581.* Proposed by *N. A. Court.*

On the line LA' joining the given point L to the centroid A' of the face BCD of a tetrahedron $ABCD$, the point P is taken such that $LP : LA'$ is equal to a given constant, positive or negative. Prove that the line AP and its three analogous lines BQ, CR, DS have a point, say U, in common.

Determine the locus of U when L describes a fixed plane.

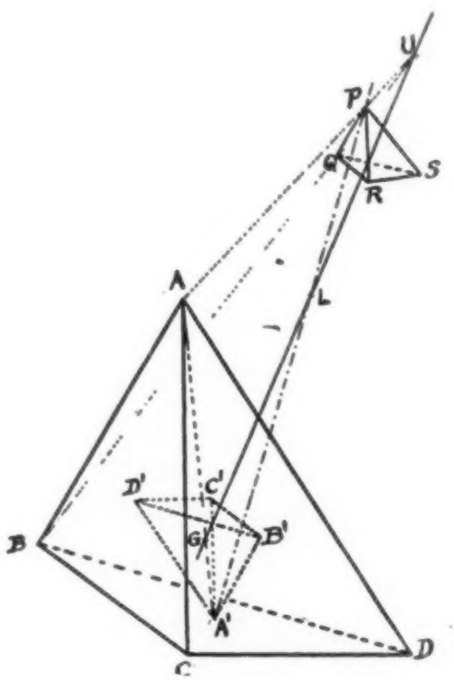
Solution by *J. S. Guérin*, Washington, D. C.

Call G the centroid of $ABCD$. The tetrahedrons $PQRS$ and $ABCD$ are homothetic to $A'B'C'D'$ with L and G as homothetic centers and, k and -3 as homothetic ratios, respectively. Therefore† $ABCD$ and $PQRS$ are homothetic, and their homothetic center, U, lies on GL.

Now the triangle LGA' cut by the transversal APU gives:

$$\frac{UL}{UG}=\frac{AA'}{AG}\cdot\frac{LP}{A'P}=\frac{4}{3}\cdot\frac{k}{k-1},$$

hence $$GL : GU=(k+3) : 3(1-k).$$



*November, 1944.

†N. A. Court, *Modern Pure Solid Geometry*, p. 18, Art. 56.

Thus as L describes a fixed plane the locus of U is the corresponding plane in an homothecy having G for center and a given ratio.

Also solved by *Howard Eves* in a similar manner. Eves shows analytically that the proposition is valid for n points in space.

BIBLIOGRAPHICAL NOTE. The corresponding problem in the plane was proposed in the *Nouvelles Annales de Mathematiques,* 1882, p. 430, Q. 1400, by Maurice D'Ocagne, the creator of Nomography.—N. A. C.

No. 586. Proposed by *N. A. Court.*

If the edges of a tetrahedron are coplanar with the polar lines for the circumsphere of the tetrahedron, of the respectively opposite edges, then the three products of the three pairs of opposite edges are equal.

I. Solution by *Howard Eves,* College of Puget Sound.

1. *Theorem.* If corresponding edges of two tetrahedra are coplanar, then the tetrahedra are in perspective.

Let the two tetrahedra be $ABCD$ and $A'B'C'D'$, with corresponding pairs of edges AB, $A'B'$; AC, $A'C'$; AD, $A'D'$; BC, $B'C'$; CD, $C'D'$; DB, $D'B'$ coplanar. Let us designate the lines AA', BB', CC', DD', by a, b, c, d. Since AB, $A'B'$ are coplanar it follows that a and b meet in some point (ab), finite or infinite. Similarly, each pair of the lines a, b, c, d meet in some point, finite or infinite. Therefore the lines a, b, c, d are either all coplanar or are concurrent. Since the former case is impossible we must have the latter, and our two tetrahedra are in perspective.

The converse of this theorem is obviously true.

2. *Theorem.* If a tetrahedron is perspective to its tangential tetrahedron, it is isodynamic.

See Art. 853 of Court's *Modern Pure Geometry.*

II. Solution by *C. E. Springer,* University of Oklahoma.

Let the four vertices $V_i(x_i, y_i, z_i)$, $(i=1,2,3,4)$, of the tetrahedron lie on the circumsphere with equation, $x^2+y^2+z^2=1$, indicated by $\sum x^2=1$. For convenience, and without loss of generality, the radius of the circumsphere is taken as unity. Any plane through the polar of line V_iV_j $(i\neq j)$ can be represented by the equation

$$m\left(\sum xx_i-1\right)+n\left(\sum xx_j-1\right)=0. \tag{1}$$

The conditions that V_k and V_l be on this plane are

$$m(\sum x_k x_i - 1) + n(\sum x_k x_j - 1) = 0$$
$$m(\sum x_l x_i - 1) + n(\sum x_l x_j - 1) = 0.$$

There exist values of m and n for which the edge $V_k V_l$ lies on the plane (1), if, and only if,

$$(2) \qquad (\sum x_k x_i)(\sum x_l x_j) - \sum x_k x_i - \sum x_l x_j + 1$$
$$= (\sum x_k x_j)(\sum x_l x_i) - \sum x_k x_j - \sum x_l x_i + 1,$$

where i, j, k, l are distinct values chosen from 1,2,3,4.

Now, by the fact that the vertices are on the sphere $\sum x^2 = 1$, the square of the product of the lengths of $V_i V_k$ and $V_l V_j$ is given by

$$(d_{ik} d_{lj})^2 = \sum (x_i - x_k)^2 \cdot \sum (x_l - x_j)^2 = (2 - 2\sum x_i x_k)$$
$$\cdot (2 - 2\sum x_l x_j) = 4[1 - \sum x_i x_k - \sum x_l x_j + (\sum x_i x_k)(\sum x_l x_j)]$$

But, by equation (2), $\qquad d_{ik} d_{lj} = d_{kj} d_{li}.$

Because i,j,k,l are distinct and take the range 1,2,3,4, the proposition follows.

No. 587 (687). Proposed by *P. D. Thomas*, U. S. Navy.

Find the equation of the ruled surface generated by a variable line which meets both the Z-axis and the curve $x = a \sin u$, $y = b \cos u$, $z = c \sin u \cos u$, and remains parallel to the XY-plane.

Solution by *A. Sisk*, Maryville, Tenn.

Let (x, y, z) be any point on the generator line PQ joining $P(a \sin u, b \cos u, c \sin u \cos u)$ to point $Q(0, 0, c \sin u \cos u)$. Then the equations for the generator PQ are

$$x/a \sin u = y/b \cos u, \quad z = c \sin u \cos u.$$

The required equation results upon eliminating the parameter u:

$$z(a^2y^2 + b^2x^2) = abcxy.$$

Also solved by *J. H. Butchart*, *Howard Eves*, *C. E. Springer*, and the *Proposer*.

No. 590. Proposed by *Nev. R. Mind.*

The midpoint of the segment joining the orthocenter of a triangle to the trace on the circumcircle of a bisector of an angle of the triangle has equal powers with respect to the two tritangent circles (i. e., circles

touching the three sides of the triangle) having their centers on the bisector considered.

Solution by *J. H. Butchart*, Grinell College.

Let I be the incenter of a triangle $(T)=ABC$ and I' the ex-center relative to the side BC. The bisector AII' of the angle A of (T) meets the circumcircle (O) in the mid-point K of the segment II', hence the projections L, M, N of K upon the sides of (T) are the mid-points of the projections of the segment II' upon those sides, therefore the powers of each of the points L, M, N for the two tritangent circles (I), (I') having I, I' for centers are equal. Thus the line LMN is the radical axis of the two circles (I), (I').

On the other hand, since K lies on the circumcircle (O) of (T), the line LMN is the Simson line of K for (T) and therefore bisects the segment KH joining K to the orthocenter H of (T) (see Court's *College Geometry*, p. 116, Art. 213). Hence the proposition.

The *Proposer*, whose solution is quite similar to the above, remarks that the proposition is valid for the external bisectors of (T). He also points out that to the two interpretations given to the line LMN and its five analogs may be added a third interpretation, namely that the six lines are the bisectors of the angles of the medial triangle of (T) (*College Geometry*, p. 170, ex. 7).

Also solved by *Howard Eves*, *Henry E. Fettis*, *J. S. Guérin*, and *Paul D. Thomas*.

PROPOSALS

No. 626. Proposed by *V. Thébault*, Tennie, Sarthe, France.

If A', B', C' are the symmetrics of the vertices of a triangle ABC with respect to a fixed point, the circumcircles of the three triangles $AB'C'$, $BC'A'$, $CA'B'$ have a point in common which lies on the circumcircle of the triangle ABC.

No. 627. Proposed by *Pedro A. Piza*, San Juan, Puerto Rico.

Show that it is possible to construct by ruler and compasses a *Fermagoric* triangle of order 3, i. e., a triangle ABC such that

$$AB^3 = AC^3 + BC^3.$$

In particular, let the line RH be divided into three segments such that $RG = s^2(r^2+rs+4s^2)$, $GF=(r^2-s^2)(r^2+rs+4s^2)$, $FH=(r+s)(4s^3 \mp r^3)$, where r, s are positive integers with $s<r<s\sqrt[3]{4}$. Further, let RH be the diameter of a circle and let AB and CD be chords perpendicular

to RH and passing through G and F respectively. Verify that this triangle ABC satisfies the above relation.

No. 628. Proposed by *W. E. Byrne*, Lexington, Virginia.

If the symbol arc tan u is defined to be the angle (in radians) between $-\pi/2$ and $\pi/2$ whose tangent is u, determine for the interval $-\pi \leqq x \leqq \pi$ the various values of the constant in the formula

$$2\gamma = \alpha + \beta + \text{constant},$$

where $\alpha = \text{arc tan}(\tfrac{1}{2}\sqrt{3} \tan x)$, $\beta = \text{arc tan}(\cos x/\sqrt{3})$

$$= \text{arc tan}\left(\frac{2 \tan \tfrac{1}{2}x + 1}{\sqrt{3}}\right).$$

(These expressions arose in a calculus problem.)

No. 629. Proposed by *D. L. MacKay*, Evander Childs High School, New York City.

Construct a triangle ABC having given, in position, the circumcenter O, the foot D of the altitude from A, and the point of intersection U of the bisector of the angle A with the side BC.

No. 630. Proposed by *P. D. Thomas*, U. S. Navy.

Projectiles are fired in a vertical plane at a given initial velocity but varying angle of elevation θ. Of all the pairs of trajectories for θ and $90° - \theta$, where $0 < \theta < 45°$, which give the same range r_θ, show that there is only one pair such that the point of maximum height attained for θ is the focus of the trajectory at $90° - \theta$. Find the value of θ for which this is true. (Consider only the flight in vacuum under the the influence of gravity).

No. 631. Proposed by *Frank C. Gentry*, University of New Mexico.

The twelve perpendiculars dropped from the four tritangent centers of a triangle (i. e. the centers of the four circles touching the three sides of the triangle) upon the sides of the triangle meet by threes in four points.

No. 632. Proposed by *V. Thébault*, Tennie, Sarthe, France.

Find a number of the form *aaabbbccc* which gives, when increased by unity, a perfect square of nine digits.

No. 633. Proposed by *N. A. Court*, University of Oklahoma.

Given two spheres (A), (B), let p be the perpendicular dropped from B upon a plane (Q) passing through A. If the traces of p and (Q) upon the plane of the common circle (s), real or imaginary, of the two spheres are pole and polar with respect to (s), the two spheres are orthogonal.

To the Contributors to the Problem Department

In order that the material received shall be more readily available for use, and that misplacement or loss may be avoided, the editors make the following suggestions:

1. Contributions should be typewritten, with double spacing and with margins of at least one inch.

2. Each page should contain one proposal, or the solution of one problem.

3. Correspondence or remarks intended for the editors should be written on separate sheets, not mixed with material intended for publication.

4. To each proposal should be prefixed a line giving the name of the proposer and the name of the institution with which he is connected or the name of the city in which he resides. (Follow the form regularly used in the MAGAZINE.)

5. Each solution should begin by reproducing the number and text of the proposal and the name of the proposer. The solution proper should be preceded by a line giving the name of the solver and his institution or city. (Follow the form regularly used in the MAGAZINE.)

6. It is important that ideas expressed shall be not only correct, but also concise and clear, and presented in good form. Attentive reading of proposals and solutions published in the MAGAZINE should provide helpful hints.

Authors should cultivate the ambition of submitting their contributions in such form that they may be published without rewriting or retouching by the editors. Other things being equal, contributions thus prepared are given preference in the selection of material to be included in the Problem Department.

Bibliography and Reviews

Edited by
H. A. SIMMONS and P. K. SMITH

Methods of Advanced Calculus. By Philip Franklin. McGraw-Hill Book Company, Inc., New York, 1944. xii+486 pages.

In the preface of this book, the author claims two principal objectives: "first, to refresh and improve the reader's technique in applying elementary calculus; second, to present those methods of advanced calculus which are most needed in applied mathematics."

The first of these two objectives must of necessity be in conflict with the second for the space available. Perhaps, if no review material had been included, the author would have added something on *Least Squares, Curve Fitting,* or even *Operational Calculus.* (These are about the only conceivable omissions.) The compromise arrived at was dictated no doubt by the desire expressed in the preface, to serve two groups of readers—*the engineer or applied scientist* seeking a reference or a text for self-study and *the student in advanced calculus* who needs a suitable text.

Most of the material in this book is common to the small group of texts having similar aims. An interesting and welcome exception in the present text is the last chapter on *Calculus of Variations.* Here-to-fore the undergraduate has not been over-exposed to this topic.

But related texts differ less in content than in arrangement or manner of presentation. One such innovation here is the somewhat spiral treatment given several topics. *Complex Variables* is introduced in Chapter I; then it reappears in full dress in Chapter V. *Vector Analysis* does likewise in Chapters III and VIII. One might suspect (on noting that the intervening material makes but slight use of the initial or elementary parts of the split topics), that an unnecessary discontinuity or interruption had been wrought. Not necessarily! For example, Chapter III entitled *Vectors, Curves, and Surfaces in Space,* exhibits a smooth transition from a review of *Solid Analytics* into the geometric foundations of vectors. Chapter I does very well also with *Elementary Functions for Complex Values and Taylor's Series.* Incidentally, this arrangement reduces the later chapters to normal size and allows some re-associations there likewise.

Quoting again from the preface, we have "Each chapter is followed by a number of problems, arranged in an order corresponding to the development of the text. There is a large number, averaging nearly 100 per chapter and they range from routine exercises to elaborate applications to science and engineering." There's more to this problem material than just that. A student who reads the text alone, omitting the problems, can acquire at best only a "reading knowledge" of the subject. But let him master these problems, and something on an entirely different level is gained, namely, a "working knowledge". Furthermore, those elaborate applications in some cases are equivalent to whole pages of text in that they introduce new material or prove additional theorems.

Some might prefer that the problems be distributed throughout the chapter, as in elementary texts. On this level it would not be unreasonable to ask the student to turn to the back of the book or even to a separately bound cover for such material.

The author refers to the text as a stepping-stone, and such it is. That means an introduction to many techniques but not an exhaustive treatment of any one topic.

The many references throughout the body of the text and the rather extensive bibliography in the back are adequate for the student taking the next step.

The selection, arrangement and development of material in this book are such as to form a digestible and palatable blending of rigor with intuition, abstract elegance with practical application. It is a distinct contribution to the field.

Alabama Polytechnic Institute. RALPH D. DONER.

Theory of Functions. Part I: Foundations of the General Theory of Analytic Functions. By Konrad Knopp. Dover Publications, New York, 1945. vii+146 pages. $1.25.

This American edition is a translation of the German fifth edition (Sammlung Goschen) by Frederick Bagemihl of the University of Rochester.

It gives a concise presentation of the usual topics covered in a beginning course in the theory of functions of a complex variable: definition of an analytic function, the Cauchy integral, series (of analytic functions, Taylor and Laurent expansions), analytic continuation, entire transcendental functions, singularities and the residue theorem.

Students who do not read German easily will find this book extremely useful for review and reference. Furthermore, the German edition is not available at present. It is understood that Part II is now being translated. There are also two companion volumes in German, which give problems and solutions related to Parts I, II of the theory.

Virginia Military Institute. W. E. BYRNE.

Consider the Calendar. By Bhola D. Panth. Bureau of Publications, Columbia University, New York, 1944. 138 pages.

In speaking of the calendar in the first chapter of this treatise the author states: "As it now stands, it is like a clock which ticks off the days, weeks, months, years, and centuries. It is intimately related to all our religious, social, and economic affairs. . . . Then priests took it into their safekeeping to ensure its efficacy, and pontiffs and kings used it for their own political advantage and economic domination, . . . ". This treatise sets forth very clearly the great influence the calendar has played in man's life; it sets forth very clearly the law of inertia as it functions in man's religious, social, economic, and scientific groping through the centuries. Again, over against this resistance to change and progress we see well analyzed in the treatise the inevitable evolutionary process of calendarial reform forced largely by the rapid elimination of isolation and the growing dependence of nation upon nation. The treatise demonstrates the fact that the nation with the progressive view has adjusted its calendar to meet the world demands in business and political relations.

In addition to this study in calendar change as related to man's business and social life, the technical and astronomical problems of making a calendar are set forth so that any intelligent reader may avail himself of the basic facts.

In Chapter I, entitled: "The Calendar in Perspective", man's primitive needs for a calendar are shown. It is shown how these needs grew as islolation was more and eliminated and trade became more complicated. It is shown that the sun, the moon with its phases, certain stars, and the seasons caused man to become conscious of the periodicity in nature and how this periodicity became the instrument of man's measurement of time and the basis of the calendar. The reader soon becomes conscious in his

reading that the problem of synchronizing months with seasons and dates with important holidays, or ritual days, are among the chief problems in calendar construction. The writer almost causes the reader to feel that he would like to do something about the calendar reform. On pages 14, 16, and 17 tabular analysis of the deficiency in synchronization of the Gregorian calendar is shown. Then early in his reading one becomes conscious of the need of synchronizing the number of weeks in the month with the month, the day of the week with the beginning of the month, the number of times a certain day of the week occurs in a month with the month, and the occurrence of holidays with fixed dates.

"Basic Calendar Concepts" is treated in Chapter II. In the second paragraph of this chapter we find: "Three fundamental movements determine our concepts of time. We get our diurnal and nocturnal concepts from the axial rotation of the earth; our month (synodic) concept from the revolution of the moon around the earth, and, finally, the seasonal and annual concepts from the revolution of the earth around the sun. These three factors constitute the antecedents and the bases of practically all calendars."

In this chapter the natural growth of the three types of calendars, viz., the lunar, the solar, and luni-solar, is developed in a clear and illuminating manner. It is shown that the lunar calendar served well those peoples of a nomadic or nonagricultural life; while the solar calendar was developed when it became necessary to cause the season to begin at the same dates each year. This type of calendar serves the agricultural way of life. The luni-solar calendar, as the name implies, is an instrument for preserving the old lunar calendar by intercallation to synchronize it with the solar calendar. On page 48, this intercalating principle is found to be: "Practically all present day calendars that follow the luni-solar system use the 19-year solar cycle, and during that period intercelate seven lunar months." In this chapter the historical origins of the week are given.

In Chapter III is discussed the "Basic Calendar Patterns". The bases for the use of one or more of the three types of calendars by certain peoples or cultures is exhaustively analyzed in this chapter. Especially interesting is the fact that the priesthood and pontifical authority have been preponderant influences against the adoption of the progressive calendar reforms.

Chapter IV is devoted to "Proposals and Trends in Calendar Improvement". The chapter deals primarily with the Thirteen-Month Fixed Calendar and the World Calendar (of 12 months). The advantages and disadvantages of each are given in detail; tabular analysis of each of the two types is displayed. Each is compared with the Gregorian calendar now in use by most peoples. The Thirteen-Month Fixed Calendar is praised for its great simplicity. However, the salient fact is brought out that this calendar will not likely be adopted due to its radical changes over the Gregorian type of calendar. The World Calendar is praised for its great improvement over the Gregorian calendar while retaining a large portion of the Gregorian calendar. Repeated attention is called to the fact that the World Calendar would call for fewer changes from the Gregorian calendar and the fact that it would not provoke so much resistance to a change over.

The concluding chapter is a strong discussion on the need of a calendar change based on the fact of the contraction of distance due to modern communication and ever-progressing international dependence in world trade and politics.

"Holidays of the United States" are discussed in Appendix A: "How to Determine the Day of the Week" in Appendix B; and "A Selected List of References" in Appendix B.

Louisiana Polytechnic Institute. P. K. SMITH.

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